MANAGEMENT SPONSORED
MINOR RESEARCH PROJECT

on

ON STEREOGRAPHIC SEMICIRCULAR MODELS

by

Dr. S.V.S. GIRIJA, Ph. D.

Department of Mathematics

Submitted to

The Research Committee

HINDU COLLEGE, GUNTUR

AUGUST 2020

Dr. S.V.S. GIRIJA
Professor of Mathematics
Hindu College
GUNTUR
Andhra Pradesh
Email id: svs.girija@gmail.com

DECLARATION

We hereby declare that the **Management, Hindu College, Guntur** sponsored Minor Research Project report titled **ON STEREOGRAPHIC SEMICIRCULAR MODELS** comprises of our own and original work. It has not been submitted fully or partially to any other institution or organization and is not published.

(S.V.S. GIRIJA)

CERTIFICATE

Certified that this is a genuine and bonafide work done by **Dr. S.V.S. GIRIJA,** Professor of Mathematics with the Minor Research Project titled **ON STEREOGRAPHIC SEMICIRCULAR MODELS** sanctioned by **Management, Hindu College, Guntur**.

The Principal

Hindu College, Guntur

ACKNOWLEDGEMENTS

At the outset I express my sincere gratitude to the **Management, Hindu College, Guntur** for sanctioning me a Minor Research Project on this fertile area viz., Circular Statistics and for their continued inspirational support in pursuing research work.

I am very much greatly thankful to Sri S.V.S. Somayaji, President and Sri Ch. Ramakrishna Murthy, Secretary and Correspondent of Hindu College Committee for having sponsored this project and their encouragement and support throughout this endeavor.

I am also grateful to my teachers Dr. I. Ramabhadra Sarma, Professor (Retd) of Mathematics, Acharya Nagarjuna University and Dr. A.V. Dattatreya Rao, Professor (Retd) of Statistics, Acharya Nagarjuna University for their academic advice and encouragement.

I express my gratitude to Smt V. Manikumari, Principal of the College, the colleagues of the Department of Mathematics for their constant help and encouragement.

S.V.S. GIRIJA

CONTENTS

CHAPTER 1

Introduction

1.1 Introduction

In some of the cases the directional / angular data does not require full circular models for modeling, this fact is noted in Guardiola (2004), Jones (1968) and Byoung et al (2008). For example, when sea turtles emerge from the ocean in search of a nesting site on dry land, a random variable having values on a semicircle is well sufficient for modeling such data. Similarly, when an aircraft is lost but its departure and its initial headings are known, a semicircular random variable is sufficient for such angular data. And few more examples of semicircular data is available in Ugai et al (1977).

Guardiola (2004) obtained the Semicircular Normal distribution by using a simple projection and Byoung et al (2008) developed a family of the semicircular Laplace distributions for modeling semicircular data by simple projection, Phani et al (2013) constructed some semicircular distributions by applying Inverse Stereographic projection. In this work we develop the **Stereographic Semicircular Half Logistic distribution** (**SSCHLD**), **Stereographic Semicircular Quasi Lindley distribution** (**SSCQLD**) and **Stereographic Semicircular Erlang distribution** (**SSCHLD**) by projecting Half Logistic distribution over a semicircular segment. Furthermore we consider the asymptotic behavior of the Stereographic Semicircular Half Logistic distribution, derive the first two trigonometric moments for proposed model, we state and prove some theorems that characterize the Stereographic Semicircular Half Logistic model. At the end of this paper we develop the Stereographic- l-axial Half Logistic distribution by extending the Stereographic Semicircular Half Logistic distribution for modeling axial data.

1.2 Mathematical and Statistical Preliminaries

(1) Bilinear Transformations / Möbius Transformation

A transformation of the form $w = T(z) = \dfrac{az+b}{cz+d}$, (1.2.1) where $a, b, c,$ and d are complex constants and $ad - bc \neq 0$ is known as Bilinear transformation or Linear fractional transformation or Möbius transformation (Ahlfors,1966; Boas,1987; Fisher, 1986; Hille, 1959).

The transformation $w = T(z) = \dfrac{az+b}{cz+d}$ can be written as $cwz + dw - az - b = 0$ which is linear both in z and w. For this reason, a linear fractional transformation is often called a bilinear transformation.

When $c = 0$, we have a linear transformation, and for $a = d = 0,\ b = c$, we have an inversion.

The condition $ad - bc \neq 0$ ensures that the mapping is not constant. i.e., $\dfrac{dw}{dz} \neq 0$, i.e., the transformation is conformal. For this, suppose that $ad - bc = 0$.

If $c \neq 0$, we have $b = \dfrac{ad}{c}$ and transformation (1.2.1) can be written as

$$T(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) + b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d} = \frac{a}{c}$$, a constant.

If $a \neq 0$, then $ad - bc = 0$ gives $d = \dfrac{bc}{a}$ so that (1.2.1) can be written as

$$T(z) = \frac{az+b}{cz + \frac{bc}{c}} = \frac{a}{c}\left(\frac{z + \frac{b}{c}}{z + \frac{b}{c}}\right) = \frac{a}{c}$$, a constant.

If $ad - bc = 0$ and $a = 0$, then either $b = 0$ or $c = 0$ .when $a = b = 0$, it follows that $w = 0$, and when $a = c = 0$,we have $w = \dfrac{b}{d}$, a constant.

Here onwards we assume that $ad - bc \neq 0$. Thus, we write

$$w = T(z) = \frac{az+b}{cz+d} = \begin{cases} \frac{a}{c} - \left(\frac{ad-bc}{c^2}\right)\frac{1}{\left(z+\frac{d}{c}\right)} & if \;\; c \neq 0 \\ \left(\frac{a}{d}\right)z + \frac{b}{d} & if \;\; c = 0 \end{cases} \tag{1.2.2}$$

The domain of the definition of $T(z)$ is $\mathbb{C} - \left\{-\frac{d}{c}\right\}$. Clearly, $T(z)$ is one -to-one function on its domain, since $T(z)$ is well defined for all points in the extended complex plane except at $z = -\frac{d}{c}$ and the point at ∞, we may extend the definition of $T(z)$ to the extended complex plane by including these points.

For $c \neq 0$, we may define $w = T(z) = \begin{cases} \frac{az+b}{cz+d} & if \;\; z \neq -\frac{d}{c}, \; z \neq \infty \\ \infty & if \;\; z = -\frac{d}{c} \\ \frac{a}{d} & if \;\; z = \infty \end{cases}$ (1.2.3)

and $T(z)$ defined in this way is then one-one and onto function on the extended complex plane and has an inverse that is also a bilinear transformation, defined as

$$z = T^{-1}(w) = \begin{cases} \frac{dw-b}{-cw+a} & if \;\; w \neq \frac{a}{c}, \; w \neq \infty \\ \infty & if \;\; w = \frac{a}{c} \\ \frac{-d}{c} & if \;\; w = \infty \end{cases} \tag{1.2.4}$$

(2) Properties of Bilinear transformation: A bilinear transformation

i. is a conformal mapping i.e., preserves angles
ii. maps circles and straight lines onto circles as well as straight lines.
iii. preserves cross-ratio of four points.

iv. cannot have more than two fixed points unless it is the identity transformation.

Also, the set of bilinear transformations forms a group under composition of mappings.

(3) Bilinear transformation is a mapping which maps points on the unit circle in the complex plane into the point *x* on the real line. [Minh and Farnum (2003)]

Let $w = T(z) = \frac{az+b}{cz+d}$ be a bilinear transformation, which maps the complex plane into itself, where $a, b, c,$ and d are constants, real or complex (Ahlfors, 1966).

i) When $c = 0$, $T(z) = \frac{az+b}{d} = \frac{a}{d} z + \frac{b}{d}$ is simply a rotation followed by a translation, which does not take a circle into a straight line.

From now onwards, we assume $c \neq 0$

ii) When $\text{Im}(c) = 0$ then $T(z)$ maps the real line into itself and therefore does not map a circle into a straight line.

From now on wards, we assume that $\text{Im}(c) \neq 0$, now we impose some constraints on the constants $a, b, c,$ and, d in order to map a unit circle $|z| = 1$ or $z = e^{i\theta}$ onto the real line.

Let $w = T(z) = \frac{az+b}{cz+d}$ and $\text{Im}(c) \neq 0$

$T(z)$ is real if and only if $\overline{T(z)}$

$$\Leftrightarrow \left(\frac{az+b}{cz+d}\right) = \overline{\left(\frac{az+b}{cz+d}\right)} \quad \Leftrightarrow \left(\frac{az+b}{cz+d}\right) = \left(\frac{\overline{az+b}}{\overline{cz+d}}\right)$$

$$\Leftrightarrow \left(\frac{az+b}{cz+d}\right) = \left(\frac{\overline{az}+\overline{b}}{\overline{cz}+\overline{d}}\right)$$

$$\Leftrightarrow \left(\frac{az+b}{cz+d}\right)=\left(\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) \quad \Leftrightarrow (az+b)\left(\bar{c}\,\bar{z}+\bar{d}\right)=(cz+d)\left(\bar{a}\,\bar{z}+\bar{b}\right)$$

$$\Leftrightarrow a\bar{c}\,z\,\bar{z}+b\bar{c}\,\bar{z}+a\bar{d}\,z+b\bar{d}=\bar{a}\;c\,z\,\bar{z}+\bar{a}\,d\,\bar{z}+\bar{b}\,c\,z+b\bar{d}$$

$$\Leftrightarrow \left(a\bar{c}-\bar{a}c\right)|z|^2+\left(a\bar{d}-\bar{b}c\right)z+\left(b\bar{c}-\bar{a}d\right)\bar{z}+\left(b\bar{d}-\bar{b}d\right)=0 \qquad (1.2.5)$$

Case (i)

If $a\bar{c}$ is real, then $\overline{a\bar{c}}=a\bar{c}$

$$\Rightarrow \bar{a}c=a\bar{c}$$

$$\Rightarrow \bar{a}c-a\bar{c}=0$$

Therefore (1.2.5) represents a straight line.

Case (ii)

If $a\bar{c}$ is not real, then $a\bar{c}-\bar{a}c\neq 0$

by dividing equation (1.2.5) with $\left(a\bar{c}-\bar{a}c\right)$, we get

$$|z|^2+\left(\frac{a\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}\right)z+\left(\frac{b\bar{c}-\bar{a}d}{a\bar{c}-\bar{a}c}\right)\bar{z}+\left(\frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}\right)=0 \qquad (1.2.6)$$

Put $\gamma=\dfrac{b\bar{c}-\bar{a}d}{a\bar{c}-\bar{a}c}$ so that $\bar{\gamma}=\dfrac{\bar{b}c-a\bar{d}}{\bar{a}c-a\bar{c}}$ $=$ $\dfrac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}c}$ and $\delta=-\left(\dfrac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}\right)$, substituting these values in equation (1.2.6) , we have $|z|^2+\bar{\gamma}z+\gamma\bar{z}-\delta=0$ (1.2.7)

Adding $|\gamma|^2$ on both sides of the equation (1.2.7) we have

$$|z|^2+|\gamma|^2+\bar{\gamma}z+\gamma\bar{z}-\delta=|\gamma|^2$$

$$\Rightarrow |z|^2+|\gamma|^2+\bar{\gamma}z+\gamma\bar{z}=\delta+|\gamma|^2$$

$$\Rightarrow |z+\gamma|^2=\delta+|\gamma|^2=\left|\frac{ad-bc}{\bar{a}c-\bar{c}a}\right|^2$$

$$\Rightarrow\left|z+\left(\frac{b\bar{c}-a\bar{d}}{a\bar{c}-\bar{a}c}\right)\right|^2=\left|\frac{ad-bc}{a\bar{c}-\bar{c}a}\right|^2$$

$$\Rightarrow\left|z+\left(\frac{b\bar{c}-a\bar{d}}{a\bar{c}-\bar{a}c}\right)\right|=\left|\frac{ad-bc}{a\bar{c}-\bar{c}a}\right| \quad ,$$

which represents a unit circle centered at the origin if and only if

(i) $\frac{b\bar{c}-\bar{a}d}{a\bar{c}-\bar{a}c}=0$ and (ii) $\left|\frac{ad-bc}{a\bar{c}-\bar{c}a}\right|=1$ (Ahlfors, 1966; p. 79), the first condition shows that

$\bar{a}d=b\bar{c}$ and $a\neq 0$. (1.2.8)

Dividing all the coefficients in equation (1.2.1) by a, we have a transformation of the form

$T(z)=\frac{z+b}{cz+d}$ (1.2.9)

In this form condition (1.2.8) becomes $d=\bar{c}b$. Finally, we impose the requirement that $T(-1)=\infty$

,

This is satisfied when $c=d$, combining this requirement with $d=\bar{c}b$, we have $b=\frac{c}{\bar{c}}$

by putting all these conditions in equation (1.2.9), we have

$$T(z)=\frac{z+\frac{c}{\bar{c}}}{cz+d}=\frac{\left(\frac{1}{c}\right)z+\frac{1}{\bar{c}}}{z+1} \qquad (1.2.10)$$

putting $C=\frac{1}{c}$ and $\bar{C}=\frac{1}{\bar{c}}$ in (1.2.10), we have

$T(z)=\frac{Cz+\bar{C}}{z+1}$ with $\operatorname{Im}(C)\neq 0$, where $C=u-iv$ and $\bar{C}=u+iv$,with $v\neq 0$ (1.2.11)

The bilinear transformation defined by (1.2.11) is a real-valued for any z on the unit circle.

Let $z=e^{i\theta}=\cos\theta+i\sin\theta$ be any point on the unit circle, now we show that

$T(z)=T(e^{i\theta})=T(\cos\theta+i\sin\theta)=x$, a real number on the real line.

$$x=T(z)=T(e^{i\theta})$$

$$=\frac{C(e^{i\theta})+\overline{C}}{e^{i\theta}+1}$$

$$=\left(\frac{C(e^{i\theta})+\overline{C}}{e^{i\theta}+1}\right)\left(\frac{e^{-i\theta}+1}{e^{-i\theta}+1}\right)$$

$$=\frac{C+\overline{C}e^{-i\theta}+Ce^{i\theta}+\overline{C}}{2+2\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

$$=\frac{2u+u(e^{i\theta}+e^{-i\theta})+iv(e^{-i\theta}-e^{i\theta})}{2+2\cos(\theta)}$$

$$=\frac{2u+u\cos(\theta)+v\sin(\theta)}{2+2\cos(\theta)}$$

$$=u+v\left(\frac{\sin(\theta)}{1+\cos(\theta)}\right)$$

$$x=T(z)=u+v\tan\left(\frac{\theta}{2}\right), \text{ which is real.} \qquad (1.2.12)$$

Hence the bilinear transformation defined by $T(z)=\frac{Cz+\overline{C}}{z+1}$, maps every point on the unit circle onto the real line.

Form (1.2.12), we have $x=u+v\tan\left(\frac{\theta}{2}\right)$

$$\Rightarrow \quad x-u=v\tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \tan\left(\frac{\theta}{2}\right)=\frac{(x-u)}{v}$$

$$\Rightarrow \; T^{-1}(x) = \theta = 2\tan^{-1}\left(\frac{x-u}{v}\right) \qquad (1.2.13)$$

This maps every point on the real line onto a point on the unit circle.

(4) Lemma (Minh and Farnum (2003))

Consider the real valued function $x = T(\theta) = u + v\tan\left(\frac{\theta}{2}\right)$. For any complex number $c = u - iv$ with $v \neq 0$,

(a) if $v<0$, $T(\theta)$ decreases monotonically from $-\infty$ to ∞ as θ increases from $-\pi$ to π .

(b) if $v>0$, $T(\theta)$ increases monotonically from $-\infty$ to ∞ as θ increases from $-\pi$ to π .

(5) Lemma (Minh and Farnum (2003))

Let $w = \frac{(x-u)}{v}$,

(a) The inverse function of $T(\theta) = u + v\tan\left(\frac{\theta}{2}\right)$ is

$$\theta = T^{-1}(x) = 2\tan^{-1}\left(\frac{x-u}{v}\right) = 2\tan^{-1}(w)$$

(b) $\dfrac{d\theta}{dx} = \dfrac{d\left(T^{-1}(x)\right)}{dx} = \dfrac{2v}{v^2+(u-x)^2} = \dfrac{2}{v\left(1+w^2\right)}$

(6) Theorem (Minh and Farnum (2003))

(a) For $v<0$,

$$F(x) = 1 - G(\theta(x));$$

$$f(x) = -g(\theta(x))\frac{2}{v\left(1+w^2\right)}.$$

(b) For $v > 0$

$$F(x) = G(\theta(x));$$

$$f(x) = g(\theta(x)) \frac{2}{v(1+w^2)}.$$

(7) Descriptive Statistics for Directional Data

This part is concerned with basic aspects of Statistical analysis of a single sample of circular measurements $\theta_1, \theta_2, ..., \theta_n$, methods for displaying the data (sample) and sampling characteristics.

The angular raw data can be represented in two ways. They can be represented by points on the circumference of a unit circle, the same being associated to each data point as shown in figure 1.1. Alternately they can be represented by drawing the radii of the unit circle, obtained by joining the origin to the data points on the circumference as in figure 1.2.

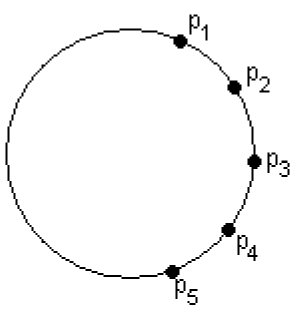


Figure 1.1

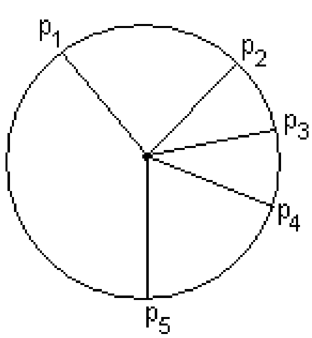


Figure 1.2

Grouped data can be represented either by linear histogram or circular Histogram or Rose diagrams or Stem and Leaf diagrams to gain an initial idea of important characteristics of the sample and suggest models for the data.

As a next step in analysis, various ways of summarizing sample information are presented briefly.

As in the linear case, the main emphasis will be on unimodal distributions. To describe unimodal circular distributions, some of the measures like mean, variance, etc. are needed. These are useful in making comparisons between unimodal distributions. It is tempting to use the conventional measures on the line (linear data), for a circular distribution. A drawback of such measures can be seen by considering an extreme example. Let us assume that the observed angles in a sample of size 2 are 4^0 and 356^0. The arithmetic mean and the sample variance give absurd results. Although intuitively we infer that the mean directions in some sense is 0^0 and deviation about the mean is roughly 1^0. We get a sensible answer by selecting the zero direction as the y-axis in place of x-axis, since the data then reduce to 266^0 and 274^0. Hence the usual linear statistical tools depend heavily on the choice of zero direction and are therefore inappropriate for circular distributions.

Mean direction

Let P_i be a point on the circumference of the unit circle corresponding to the angle $\theta_i, i = 1(1)n$. Then the mean direction θ_0 of $\theta_1, \theta_2, ..., \theta_n$ is defined to be the direction of the resultant of the unit vectors $\overline{OP}_1, \overline{OP}_2, ..., \overline{OP}_n$. The Cartesian coordinates of P_i are $(\cos\theta_i, \sin\theta_i)$ so that the centre of gravity of these points is (C, S) where

$$C = \frac{1}{n}\sum \cos\theta_i \quad \text{and} \quad S = \frac{1}{n}\sum \sin\theta_i$$

then $R = \sqrt{C^2 + S^2}$

gives the length of the resultant vector. The direction θ_0 of the resultant vector,

$$\theta_0 = \begin{cases} \tan^{-1}(S/C) & \text{if } \ S > 0,\ C > 0 \\ \tan^{-1}(S/C) + \pi & \text{if } \ C < 0 \\ \tan^{-1}(S/C) + 2\pi & \text{if } \ S < 0,\ C > 0 \end{cases}$$

is called the mean direction. Since the sample observations are subject to random fluctuations, various sample characteristics such as mean direction, length of the resultant vector, etc., deviate to some extent from those of the parent population.

It may be noted that θ_0 is rotationally equivariant (if the data is shifted by a certain amount, the value of θ_0 also changes by the same amount) and similarly θ_0 is equivariant w.r.t. changes in the sense of rotation (when one switches from clockwise to anticlockwise θ_i's become $(2\pi - \theta_i)$ and θ_0 will be $(2\pi - \theta_0)$).

Circular distance and Measure of Dispersion

The direction of the resultant vector provides mean direction and its length is a useful measure (for unimodal data) of 'how concentrated' the data is towards this centre. If all the *n* observations (unit vectors) are pointing in the same direction indicating large concentrations, then its length can be as large as '*n*'. Conversely, if the data is evenly spread over the circle indicating 'no concentration', its length can be as small as zero. Therefore, the length of the resultant vector lies between (0, *n*). Hence $\overline{R}$ **the mean resultant length** associated with mean vector (=*R/n*) lies in the range (0,1). Therefore, an appropriate measure of 'Distance' on the circle (*n-R*) is indeed the right analogue of the usual sample variance. Other noteworthy measures of distance are,

Distance between any two points A, B on the circle

Let A and B be any two points on the unit circle with respective angles θ_1 and θ_2 as shown in figure 1.3. A reasonable measure of circular distance between A and B is to consider the 'smaller of the two arc lengths' between the points along the circumference.

Then the distance

d(A,B) = min $[(\theta_1 - \theta_2),\ 2\pi - (\theta_1 - \theta_2)] = \pi - |\pi - (\theta_1 - \theta_2)|$.

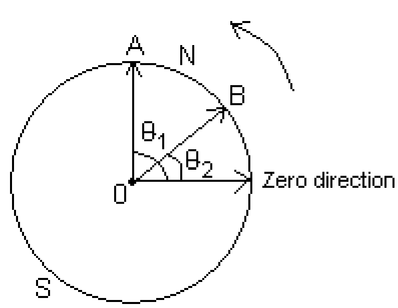


Figure 1.3

From the figure 1.3, it may be noted that the distance between A and B can be the length ANB or ASB. But, since the length ANB is shorter, we take it to be the circular distance. Clearly no two points on the circumference of a circle can be farther than π i.e. the circular distance lies in $[0, \pi]$. Alternatively the distance can be taken as

$$d(\theta_1, \theta_2) = [1 - \cos(\theta_1 - \theta_2)]$$

which is a monotonically increasing function of ($\theta_1 - \theta_2$) and lies in the interval [0, 2].

Circular distance between any point to several points.

If θ_i denotes the angle between u_i and the arbitrary point (vector) V $(0 \le \theta_i \le \pi)$ as shown in figure 1.4, then the circular distance can be defined as $d(V, u_i) = [1 - \cos\theta_i]$.

Using this, the sample dispersion w.r.t. the arbitrary vector *V*, denoted by D_V, can be defined as

$$D_V(u_i, u_2, ..., u_n) = \sum_{i=1}^{n} d(V, u_i) = n - \sum_{i=1}^{n} \cos\theta_i$$

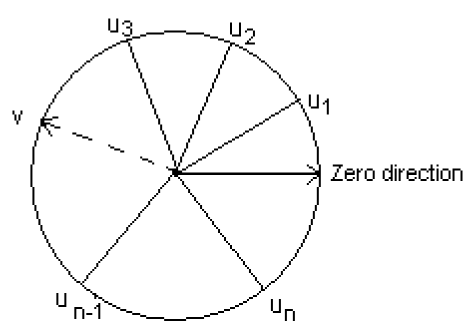


Figure 1.4

It can be observed that D_V attains its minimum when *V* is the normalized resultant vector $V^* = \left(\frac{\sum \cos\theta_i}{R}, \frac{\sum \sin\theta_i}{R} \right)$

(8) Circular Distribution [Jammalamadaka and Sen Gupta (2001)]

In the continuous case $g:[0,2\pi)\to \mathbb{R}$ is the probability density function of a circular distribution if and only if *g* has the following basic properties

- $g(\theta) \geq 0, \ \forall\theta$ (1.2.14)
- $\int_0^{2\pi} g(\theta)\, d\theta = 1$ (1.2.15)
- $g(\theta) = g(\theta + 2k\pi)$ (1.2.16)

  for any integer k (i.e., g is periodic) (Mardia,1972)

It may be noted that the circular distribution is a probability distribution whose total probability is concentrated on the unit circle $\{(\cos\theta, \sin\theta)/ 0 \leq \theta < 2\pi \}$ in the plane which satisfies the properties (1.2.14) through (1.2.16).

If $G(\theta)$ denotes the cdf of the r.v., the characteristic function of the circular model is given by

$$\phi_\theta(t) = E\left(e^{it\theta}\right) = \int_0^{2\pi} e^{it\theta} dG(\theta) = \rho_t e^{i\mu_t} \quad t \in \mathbb{R}$$

It can be seen that whenever $\varphi(t) \neq 0$, $e^{2\pi it} = 1$ (Mardia 1972 p. 41). This suggests that the function $\varphi(t)$ should only be defined for integer values of *t*. Accordingly the characteristic function $\varphi(p) = \varphi_p$ is defined by

$$\varphi_\theta(p) = E\left(e^{ip\theta}\right) = \int_0^{2\pi} e^{ip\theta} dF(\theta) = \rho_p e^{i\mu_p} \qquad p \in \mathbb{Z}$$

Clearly, $\varphi_0 = 1, \quad \overline{\varphi}_p = \varphi_{-p}$.

(9) Trigonometric moments [Jammalamadaka and Sen Gupta (2001)]

The value of the characteristic function φ_p at an integer p is also called the p^{th} trigonometric moment of θ. The real and the imaginary parts of φ_p are denoted by α_p and β_p respectively. We can also view these trigonometric moments in terms of

$$\alpha_p = E(\cos p\theta), \quad \beta_p = E(\sin p\theta), \quad p \in \mathbb{Z}$$

The first trigonometric moment namely, $\varphi_1 = \alpha_1 + i\beta_1 = \rho_1 e^{i\mu_1}$ plays a prominent role in determining the mean direction and resultant length.

The pdf of a wrapped circular model can be obtained through the characteristic function of the linear r.v. X using trigonometric moments. Using the inversion theorem of the characteristic function, one can derive circular models through trigonometric moments. These trigonometric moments can be obtained using the following Proposition [Jammalamadaka and Sen Gupta (2001), p.31] and Carslaw (1930).

Proposition *The trigonometric moment of order* p *for a wrapped circular distribution corresponds to the value of the characteristic function of the unwrapped r.v.* X*, say* $\varphi_p = \phi_X(p)$ *for* $p \in \mathbb{Z}$.

If α_p and β_p are the trigonometric moments and $\sum_{p=1}^{\infty}\left(\alpha_p^2 + \beta_p^2\right)$ is convergent then the random variable θ has a density g which is defined **almost everywhere** by

$$g(\theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\phi_p e^{-ip\theta}$$

$$= \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}(\alpha_p \cos p\theta + \beta_p \sin p\theta)\right], \theta \in [0, 2\pi), p \in \mathbb{Z} \qquad (1.2.17)$$

where the convergence of the sum is in the L^2 sense:

$$\lim_{n} \int_{0}^{2\pi} \left| g(\theta) - 1 - 2\sum_{p=1}^{n} \left(\alpha_p \cos p\theta + \beta_p \sin p\theta\right) \right|^2 d\theta = 0$$

From the uniqueness theorem, the distribution function G on the circle is uniquely determined by the characteristic function [Mardia (1972) c.f. p. 80]. The characteristic function and the cdf of the resultant circular model respectively are,

$$\phi_W(p) = E(e^{ip\theta}) = \phi(p),\ p \in \mathbb{Z}$$

and $G(\theta) = \int_{0}^{\theta} g(\theta') d\theta', \theta \in [0, 2\pi)$

(10) Population Characteristics of Circular Models

Mardia and Jupp (2000) gave expressions of ρ_p, μ_o, ρ_1, V_o, σ_o, α_p^*, β_p^*, γ_1^o and γ_2^o for circular distributions. These characteristics for the new circular models are also based on their respective trigonometric moments. These can be expressed in terms of trigonometric moments α_p and β_p .

The mean direction and the resultant length

The mean direction is given by

$$\mu_o = \mu_1^o = \tan^{-1}\left(\frac{\beta_1}{\alpha_1}\right) \tag{1.2.18}$$

In particular, the resultant length is

$$\rho = \rho_1 = \sqrt{\alpha_1^2 + \beta_1^2} \tag{1.2.19}$$

Circular Variance and Standard deviation

The circular variance is $V_O = 1 - \rho$ (1.2.20)

and the standard deviation is

$$\sigma_o = \sqrt{-2\ln(1-V_o)} \tag{1.2.21}$$

Central Trigonometric Moments

The central trigonometric moments of a circular random variable X_S are defined by the equation

$$\mu_p = Ee^{ip(X_S-\mu_0)} = \alpha_p^* + i\beta_p^* \tag{1.2.22}$$

where μ_0 is the mean direction . Since the expectation in the above is given by $\phi_p e^{-ip\mu_0}$, the central trigonometric moments are

$$\left.\begin{aligned} \alpha_p^* &= \rho_p \cos\left(\mu_p^o - p\mu_o\right), \\ \beta_p^* &= \rho_p \sin\left(\mu_p^o - p\mu_o\right) \end{aligned}\right\} \tag{1.2.23}$$

Skewness and Kurtosis

To calculate skewness and kurtosis we need the second trigonometric moments of circular random variable X_S. One common measure of skewness γ_1^o of a circular distribution is

$$\gamma_1^o = \beta_2^* / V_o^{3/2}$$

$$= \frac{\beta_2\left(\rho^2 - 2\beta_1^2\right) - 2\alpha_1\alpha_2\beta_1}{\rho^2(1-\rho)^{3/2}} \tag{1.2.24}$$

Next, we consider the circular kurtosis,

$$\gamma_2^o = \frac{\alpha_2^* - (1-V_o)^4}{V_o^2} \tag{1.2.25}$$

1.3 Methodology of Modified Inverse Stereographic Projection

Modified Inverse Stereographic Projection is defined by a one to one mapping given by $T(\theta) = x = v\tan\left(\frac{\theta}{2}\right)$, where $x \in (-\infty, \infty)$, $\theta \in [-\pi, \pi)$ and $v \in R^{+}$. Suppose x is randomly chosen on the interval $(-\infty, \infty)$. Let $F(x)$ and $f(x)$ denote the cumulative distribution and probability density functions of the random variable X respectively. Then $T^{-1}(x) = \theta = 2\tan^{-1}\left(\frac{x}{v}\right)$ by Toshihiro Abe et al (2010) is a random point on the unit circle. Let $G(\theta)$ and $g(\theta)$ denote the cumulative distribution and probability density functions of this random point θ respectively. Then $G(\theta)$ and $g(\theta)$ can be written in terms of $F(x)$ and $f(x)$ using the following Theorem.

Theorem 1.1: For $v > 0$,

i) $$G(\theta) = F\left(v\tan\left(\frac{\theta}{2}\right)\right) = F(x(\theta))$$

ii) $$g(\theta) = v\left(\frac{1+\tan^{2}\left(\frac{\theta}{2}\right)}{2}\right) f\left(v\tan\left(\frac{\theta}{2}\right)\right)$$

If a linear random variable X has a support on R, then θ has a support on $(-\pi, \pi)$ and if X has a support on R^{+}, then θ has a support on $(0, \pi)$. These means that, after the Inverse stereographic projection is applied, we can deal circular data if the support of X is on R and we can handle semicircular data if the support of X is on R^{+}.

CHAPTER 2

Stereographic Semicircular Half Logistic distribution

2.1 Stereographic Semicircular Half Logistic distribution

A random variable X on the real line is said to have an Half Logistic distribution with location parameter α and scale parameter $\beta > 0,$ if the probability density function and probability distribution function of X are given respectively by

$$f(x) = \frac{2}{\beta}\left[1+\exp\left(\frac{-(x-\alpha)}{\beta}\right)\right]^{-2}\exp\left(\frac{-(x-\alpha)}{\beta}\right), \qquad 0 < x < \infty,\ \beta > 0. \tag{2.1.1}$$

$$F(x) = \frac{1-\exp\left(\frac{-(x-\alpha)}{\beta}\right)}{1+\exp\left(\frac{-(x-\alpha)}{\beta}\right)}, \quad 0 < x < \infty \tag{2.1.2}$$

Then by applying modified inverse Stereographic projection defined by a one to one mapping $x = v\tan\left(\frac{\theta}{2}\right), v \in \mathbb{R}^{+},$ this leads to a Semicircular Model on unit semicircle. We call this model as Stereographic Semicircular Half Logistic Distribution.

Definition:

A random variable X_{SC} on the semicircle is said to have the Stereographic Semicircular Half Logistic distribution with location parameter μ scale parameter $\sigma > 0$ denoted by S**SCLD** (σ, μ), if the probability density and the cumulative distribution functions are respectively given by

$$g(\theta)=\frac{1}{\sigma}\sec^2\left(\frac{\theta}{2}\right)\left[1+\exp\left(\frac{-\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right]^{-2}\exp\left(-\left(\frac{\tan\left(\frac{\theta}{2}\right)-\mu}{\sigma}\right)\right), \tag{2.1.3}$$

$$0<\theta<\pi,\ \sigma=\frac{\beta}{\nu}>0,\ \mu=\frac{\alpha}{\nu}$$

$$G(\theta)=\left[1-\exp\left(\frac{-\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right]\left[1+\exp\left(\frac{-\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right]^{-1},$$

$$0<\theta<\pi,\ \sigma=\frac{\beta}{\nu}>0,\ \mu=\frac{\alpha}{\nu} \tag{2.1.4}$$

Hence the proposed new model S**SCHLD** (σ,μ) is a semicircular model.

We consider the asymptotic behavior of the Stereographic Semicircular Half Logistic distribution when $\sigma\to 0$. Suppose θ follows S**SCHLD** (σ,μ) **.** Let $y=\frac{\theta}{\sigma}$ and then use the change of variable technique. For sufficiently small $\sigma,$ we have $\tan\left(\frac{\sigma y}{2}\right)\approx\frac{\sigma y}{2}$ and $\sec\left(\frac{\sigma y}{2}\right)\approx 1$ by first order approximation of the Taylor series expansion. Hence, the distribution of θ becomes Half Logistic distribution (linear). So, for sufficiently small $\sigma,$ the Stereographic Semicircular Half Logistic distribution can be approximated by a linear Half Logistic distribution.

2.2. Trigonometric moments of the Stereographic Semicircular Half Logistic distribution

The trigonometric moments of the distribution are given by $\{\varphi_{\mathrm{p}}:\mathrm{p}=\pm1,\pm2,\pm3,...\}$, where $\varphi_{\mathrm{p}}=\alpha_{\mathrm{p}}+i\beta_{\mathrm{p}},$ with $\alpha_{\mathrm{p}}=E(\cos\mathrm{p}\theta)$ and $\beta_{\mathrm{p}}=E(\sin\mathrm{p}\theta)$ being the p^{th} order cosine and sine moments of the random angle θ respectively.

Theorem 2.1 With the probability density function of Stereographic Semicircular Half Logistic distribution with $\mu=0$, the first two trigonometric moments are

$$\alpha_1 = 1-\frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(\mathrm{n}+1)G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}-\frac{1}{2}\\ -\frac{1}{2},0,\frac{1}{2}\end{matrix}\right),$$

$$\beta_1 = \frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(\mathrm{n}+1)G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}0\\ 0,0,\frac{1}{2}\end{matrix}\right),$$

$$\alpha_2 = 1+\frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(\mathrm{n}+1)\left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}-\frac{3}{2}\\ -\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)-G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}-\frac{1}{2}\\ -\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)\right],$$

$$\beta_2 = \frac{4}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(\mathrm{n}+1)\left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}0\\ 0,0,\frac{1}{2}\end{matrix}\right)-4G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}-1\\ 0,0,\frac{1}{2}\end{matrix}\right)\right].$$

The first two trigonometric moments are sufficient for calculating population characteristics.

Where $\int_0^{\infty} x^{2\nu-1}\left(u^2+x^2\right)^{Q-1}e^{-\mu x}dx = \frac{u^{2\nu+2Q-2}}{2\sqrt{\pi}\Gamma(1-Q)}G_{13}^{31}\left(\frac{\mu^2u^2}{4}\middle|\begin{matrix}1-\nu\\ 1-Q-\nu,0,\frac{1}{2}\end{matrix}\right)$

for $|\arg u\pi|<\frac{\pi}{2}$, $\mathrm{Re}\,\mu>0$ and $\mathrm{Re}\,\nu>0$ and $G_{13}^{31}\left(\frac{\mu^2u^2}{4}\middle|\begin{matrix}1-\nu\\ 1-Q-\nu,0,\frac{1}{2}\end{matrix}\right)$ is called as Meijer's **G-**

function (Gradshteyn and Ryzhik, 2007, formula no. 3.389.2).

Proof:

$$\varphi_{\mathrm{p}} = \int_0^{\pi} e^{ip\theta}g(\theta)d\theta = \int_0^{\pi}\cos(\mathrm{p}\,\theta)g(\theta)d\theta + i\int_0^{\pi}\sin(\mathrm{p}\,\theta)g(\theta)d\theta$$

To derive the first cosine moment $\alpha_1 = \frac{1}{\sigma}\int_0^{\pi} \cos\theta \sec^2\left(\frac{\theta}{2}\right)\left[1+e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)}\right]^{-2} e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)} d\theta$, we use the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\cos(\theta) = 1 - \frac{2x^2}{1+x^2}$ and the above integral formula

$$\alpha_1 = \frac{2}{\sigma}\int_0^{\infty}\left[1-\frac{2x^2}{1+x^2}\right]\left[1+e^{-\frac{x}{\sigma}}\right]^{-2} e^{-\frac{x}{\sigma}} dx$$

$$= 1 - \frac{4}{\sigma}\sum_{n=0}^{\infty}(-1)^n (\mathrm{n}+1)\int_0^{\infty} x^{2(\frac{3}{2})-1}(1+x^2)^{0-1} e^{-\left(\frac{\mathrm{n}+1}{\sigma}\right)x} dx$$

$$\alpha_1 = 1 - \frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n (\mathrm{n}+1) G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix}\right)$$

To derive the first sine moment $\beta_1 = \frac{1}{\sigma}\int_0^{\pi} \sin\theta \sec^2\left(\frac{\theta}{2}\right)\left[1+e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)}\right]^{-2} e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)} d\theta$, we use the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\sin(\theta) = \frac{2x}{1+x^2}$ and result follows by the same integral formula of α_1.

$$\beta_1 = \frac{4}{\sigma}\int_0^{\infty}\left[\frac{x}{1+x^2}\right]\left[1+e^{-\frac{x}{\sigma}}\right]^{-2} e^{-\frac{x}{\sigma}} dx$$

$$= \frac{4}{\sigma}\sum_{n=0}^{\infty}(-1)^n (\mathrm{n}+1)\int_0^{\infty} x^{2(1)-1}(1+x^2)^{0-1} e^{-\left(\frac{\mathrm{n}+1}{\sigma}\right)x} dx$$

$$\beta_1 = \frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n (\mathrm{n}+1) G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix}\right)$$

To obtain second cosine and sine moments α_2 and β_2, we use the transformations $x=\tan\left(\frac{\theta}{2}\right)$,

$\cos 2\theta = 1+\frac{8x^4}{\left(1+x^2\right)^2}-\frac{8x^2}{\left(1+x^2\right)}$ and $\sin 2\theta=\frac{4x}{\left(1+x^2\right)}-\frac{8x^3}{\left(1+x^2\right)^2}$, the results of α_2 and β_2 follows by the same integral formula of α_1.

$$\alpha_2 = 1+\frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(\mathrm{n}+1)\left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}-\frac{3}{2}\\ -\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)-G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}-\frac{1}{2}\\ -\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)\right],$$

$$\beta_2 = \frac{4}{\sigma\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n(\mathrm{n}+1)\left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}0\\ 0,0,\frac{1}{2}\end{matrix}\right)-4G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2\middle|\begin{matrix}-1\\ 0,0,\frac{1}{2}\end{matrix}\right)\right].$$

Higher-order moments can be obtained similarly.

2.3 Some Characterization Theorems of Stereographic Semicircular Half Logistic Distribution

Here we recall a probability density and distribution functions of Stereographic Semicircular Exponential (Phani et al (2013)) and Circular uniform distributions that will involve in the characterization theorems of Stereographic Semicircular Half Logistics Distribution.

a) Stereographic Semicircular Exponential Distribution(Phani et al (2013))

A random variable X_{SC} on unit semicircle is said to have Stereographic Semicircular Exponential distribution with scale parameter $\sigma>0$, denoted by SSEXP (σ), if the probability density and cumulative distribution functions are given by

$$g(\theta)=\frac{\sigma}{2}\sec^2\left(\frac{\theta}{2}\right)\exp\left(-\sigma\tan\left(\frac{\theta}{2}\right)\right),$$

$$\text{for } 0\le\theta<\pi \text{ and } \sigma>0 \qquad (2.3.1)$$

$$G(\theta) = 1 - e^{-\sigma \tan\left(\frac{\theta}{2}\right)} \tag{2.3.2}$$

b) Semicircular Uniform Distribution

A random variable X_C on unit semicircle is said to have Semicircular Uniform distribution, if the probability density function is given by

$$g(\theta) = \frac{1}{\pi}, \quad 0 \le \theta < \pi \tag{2.3.3}$$

Theorem 2.2

Let θ be a continuous circular random variable with probability density function $g(\theta)$. Then the random variable $\phi = 2\tan^{-1}\left(\ln\left(\frac{1+e^{-\tan\left(\frac{\theta}{2}\right)}}{1-e^{-\tan\left(\frac{\theta}{2}\right)}} \right) \right)$ follows stereographic semicircular half logistic distribution if and only if θ follows Stereographic Semicircular Exponential distribution [6] with scale parameter $\sigma = 1$.

Proof: Suppose θ follows Stereographic Semicircular Exponential distribution with scale parameter $\sigma = 1$,

$$g(\theta) = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\exp\left(-\tan\left(\frac{\theta}{2}\right)\right), \quad 0 \le \theta < \pi$$

Let $\phi = 2\tan^{-1}\left(\ln\left(\frac{1+e^{-\tan\left(\frac{\theta}{2}\right)}}{1-e^{-\tan\left(\frac{\theta}{2}\right)}} \right) \right)$, this implies that $\theta = 2\tan^{-1}\left(\ln\left(\frac{1+e^{-\tan\left(\frac{\phi}{2}\right)}}{1-e^{-\tan\left(\frac{\phi}{2}\right)}} \right) \right)$ and

$$h(\phi) = |J| \times g_\theta(\phi)$$

$$h(\phi)=\frac{\sec^2\left(\frac{\phi}{2}\right)e^{-\tan\left(\frac{\phi}{2}\right)}}{\left(1+e^{-\tan\left(\frac{\phi}{2}\right)}\right)^2},\ 0\le\phi<\pi$$, which is the density function of Stereographic Semicircular Half Logistic distribution.

Conversely, suppose that the random variable $\phi=2\tan^{-1}\left(\ln\left(\frac{1+e^{-\tan\left(\frac{\theta}{2}\right)}}{1-e^{-\tan\left(\frac{\theta}{2}\right)}}\right)\right)$ follows Stereographic Semicircular Half Logistic distribution, then the distribution function of θ is

$$G_\theta(\theta_0)=\Pr(\theta\le\theta_0)$$

$$=\Pr\left(2\tan^{-1}\left(\ln\left(\frac{1+e^{-\tan\left(\frac{\phi}{2}\right)}}{1-e^{-\tan\left(\frac{\phi}{2}\right)}}\right)\right)\le\theta_0\right)$$

$$=\Pr\left(\left(\frac{1+e^{-\tan\left(\frac{\phi}{2}\right)}}{1-e^{-\tan\left(\frac{\phi}{2}\right)}}\right)\le e^{\tan\left(\frac{\theta_0}{2}\right)}\right)$$

$$=1-\Pr\left(\phi\le 2\tan^{-1}\left(\ln\left(\frac{1+e^{-\tan\left(\frac{\theta_0}{2}\right)}}{1-e^{-\tan\left(\frac{\theta_0}{2}\right)}}\right)\right)\right)$$

$G_\theta(\theta_0)=1-e^{-\tan\left(\frac{\theta_0}{2}\right)}$, which is the distribution function of the Stereographic Semicircular Exponential distribution with scale parameter $\sigma=1$.

Hence the theorem.

Theorem 2.3

Let θ be a continuous circular random variable with probability density function $g(\theta)$. Then the random variable $\phi=2\tan^{-1}\left(\ln\left(2e^{\tan\left(\frac{\theta}{2}\right)}-1\right)\right)$ follows Stereographic Semicircular Half Logistic distribution if and only if θ follows Stereographic Semicircular Exponential distribution with scale parameter $\sigma=1$.

Proof:

Suppose θ follows Stereographic Semicircular Exponential distribution with scale parameter $\sigma = 1$,

$$g(\theta) = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\exp\left(-\tan\left(\frac{\theta}{2}\right)\right), \quad 0 \le \theta < \pi$$

Let $\phi = 2\tan^{-1}\left(\ln\left(2e^{\tan\left(\frac{\theta}{2}\right)} - 1\right)\right)$, this implies that $\theta = 2\tan^{-1}\left(\ln\left(\frac{1+e^{\tan\left(\frac{\phi}{2}\right)}}{2}\right)\right)$ and the Jacobian of this transformation is $|J| = \frac{\sec^2\left(\frac{\phi}{2}\right)e^{\tan\left(\frac{\phi}{2}\right)}}{2\left(1+e^{\tan\left(\frac{\phi}{2}\right)}\right)}$.

Therefore, $h(\phi) = |J| \times g_\theta(\phi)$

$$h(\phi) = \frac{\sec^2\left(\frac{\phi}{2}\right)e^{-\tan\left(\frac{\phi}{2}\right)}}{\left(1+e^{-\tan\left(\frac{\phi}{2}\right)}\right)^2}, \quad 0 \le \phi < \pi$$, which is the density function of Stereographic Semicircular Half Logistic distribution.

Conversely, suppose that the random variable $\phi = 2\tan^{-1}\left(\ln\left(2e^{\tan\left(\frac{\theta}{2}\right)} - 1\right)\right)$ follows Stereographic Semicircular Half Logistic distribution, then the distribution function of θ is

$$G_\theta(\theta_0) = \Pr(\theta \le \theta_0)$$

$$= \Pr\left(2\tan^{-1}\left(\ln\left(\frac{1+e^{\tan\left(\frac{\phi}{2}\right)}}{2}\right)\right) \le \theta_0\right)$$

$$= \Pr\left(\phi \le 2\tan^{-1}\left(\ln\left(2e^{\tan\left(\frac{\theta_0}{2}\right)} - 1\right)\right)\right)$$

$=1-e^{\tan\left(\frac{\theta_0}{2}\right)}$, which is the distribution function of the Stereographic Semicircular Exponential distribution with scale parameter $\sigma = 1$.

Hence the theorem.

Theorem 2.4

Let θ be a circular random variable which follows Semicircular uniform distribution, then the random variable $\phi = 2\tan^{-1}\left(\ln\left(\frac{p-\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta}{2}\right)}\right)\right)$ has Stereographic Semicircular Half Logistic distribution with scale parameter $\sigma = 1$ if and only if $p = \pi$.

Proof: Suppose θ follows semicircular uniform distribution with probability density function

$$g(\theta) = \frac{1}{\pi}, \; 0 \le \theta < \pi.$$

Let $\phi = 2\tan^{-1}\left(\ln\left(\frac{p-\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta}{2}\right)}\right)\right)$, this implies $\theta = 2\tan^{-1}\left(\frac{p}{1+e^{\tan\left(\frac{\theta}{2}\right)}}\right)$ and the Jacobian of the transformation is $|J| = \frac{p\sec^2\left(\frac{\theta}{2}\right)e^{\tan\left(\frac{\theta}{2}\right)}}{\left(1+e^{\tan\left(\frac{\theta}{2}\right)}\right)^2}$.

Thus the density function of ϕ is $h(\phi) = |J| \times g_\theta(\phi)$

$$h(\phi) = \frac{p\sec^2\left(\frac{\theta}{2}\right)e^{\tan\left(\frac{\theta}{2}\right)}}{\pi\left(1+e^{\tan\left(\frac{\theta}{2}\right)}\right)^2}$$

, which is the density function of stereographic semicircular half logistic distribution if $p = \pi$.

Conversely, suppose ϕ follows Stereographic Semicircular Half Logistic distribution with scale parameter $\sigma = 1$, then

$$G_\theta(\theta_0) = P\text{r}(\theta \le \theta_0)$$

$$= Pr\left(\theta \le 2\tan^{-1}\left(\frac{p}{1+e^{\tan\left(\frac{\theta}{2}\right)}}\right)\right)$$

$$= 1 - Pr\left(\phi \le 2\tan^{-1}\left(\ln\left(\frac{p-\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta}{2}\right)}\right)\right)\right) = \frac{1}{\pi} \quad \text{for } p = \pi.$$

Hence the theorem.

Theorem 2.5

The circular random variable θ follows Stereographic Semicircular Half Logistic distribution with scale parameter $\sigma = 1$ with density function $g(\theta)$ if and only if $g(\theta)$ satisfies the Initial value problem $\frac{dg}{d\theta} - \left(\tan\left(\frac{\theta}{2}\right) + \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\right)g = \left(1+e^{\tan\left(\frac{\theta}{2}\right)}\right)g^2$, with g(0)= $\frac{1}{4}$.

Proof: Consider the differential equation

$$\frac{dg}{d\theta} - \left(\tan\left(\frac{\theta}{2}\right) + \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\right)g = \left(1+e^{\tan\left(\frac{\theta}{2}\right)}\right)g^2, \text{ with } g(0)=\frac{1}{4}. \tag{2.3.4}$$

Suppose θ follows Stereographic Semicircular Half Logistic distribution with scale parameter $\sigma = 1$.

Its density function is $g(\theta) = \sec^2\left(\frac{\theta}{2}\right)\left[1+\exp\left(-\left(\tan\left(\frac{\theta}{2}\right)\right)\right)\right]^{-2}\exp\left(-\left(\tan\left(\frac{\theta}{2}\right)\right)\right)$, (2.3.5)

It is easily shown that $g(\theta)$ satisfies the initial value problem (2.3.4).

Conversely, assume that $g(\theta)$ satisfies the initial value problem (2.3.4), which is a non linear differential equation (in particular Bernoulli's equation), by solving this equation, we have

$g(\theta) = \sec^2\left(\frac{\theta}{2}\right)\left(1+e^{-\tan\left(\frac{\theta}{2}\right)}\right)^{-2} e^{-\tan\left(\frac{\theta}{2}\right)}$, which is the probability density function of Stereographic Semicircular Half Logistic distribution with scale parameter $\sigma = 1$.

Hence the theorem.

Theorem 4.5

The circular random variable θ follows Stereographic Semicircular Half Logistic distribution with scale parameter $\sigma = 1$ with distribution function $\mathrm{G}(\theta)$ (2.4) if and only if $\mathrm{G}(\theta)$ satisfies the Initial value problem

$$\frac{dG}{d\theta} - \left(\frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) e^{-\tan\left(\frac{\theta}{2}\right)}}{\left(1+e^{-\tan\left(\frac{\theta}{2}\right)}\right)} \right) G = -\frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) e^{-\tan\left(\frac{\theta}{2}\right)}}{\left(1+e^{-\tan\left(\frac{\theta}{2}\right)}\right)}, \text{ with } G(\pi)=1.$$

Proof: Consider the differential equation

$$\frac{dG}{d\theta} - \left(\frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) e^{-\tan\left(\frac{\theta}{2}\right)}}{\left(1+e^{-\tan\left(\frac{\theta}{2}\right)}\right)} \right) G = -\frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) e^{-\tan\left(\frac{\theta}{2}\right)}}{\left(1+e^{-\tan\left(\frac{\theta}{2}\right)}\right)}, \text{ with } G(\pi)=1. \tag{2.3.6}$$

Suppose θ follows Stereographic Semicircular Half Logistic distribution with scale parameter $\sigma = 1$.

Its distribution function is $G(\theta) = \left[1-\exp\left(-\tan\left(\frac{\theta}{2}\right)\right)\right]\left[1+\exp\left(-\tan\left(\frac{\theta}{2}\right)\right)\right]^{-1}$, (2.3.7)

It is easily shown that $G(\theta)$ satisfies the initial value problem (2.3.4).

Conversely, assume that $G(\theta)$ satisfies the initial value problem (2.3.4), which is a non linear differential equation, by solving this equation, we have

$G(\theta)=\left[1-\exp\left(-\tan\left(\frac{\theta}{2}\right)\right)\right]\left[1+\exp\left(-\tan\left(\frac{\theta}{2}\right)\right)\right]^{-1}$, which is the distribution function of Stereographic Circular Logistic distribution with scale parameter $\sigma=1$.

Hence the theorem.

2.4 Stereographic -l-axial Half Logistic Distribution

We extend the above Stereographic Semicircular Half Logistic model to the l -axial distribution, which is applicable to any arc of arbitrary length say $2\pi/l$ for $l=1,2,...,$ so it is desirable to extend the Stereographic Semicircular Half logistic distribution to construct the Stereographic- l -axial Half Logistic distribution, we consider the density function of Stereographic Semicircular Half Logistic distribution and use the transformation $\phi=\frac{2\theta}{l}$, $l=1,2,...,$. The probability density function of ϕ is given by

$$g(\phi)=\frac{l}{2\sigma}\sec^2\left(\frac{l\phi}{4}\right)\left[1+\exp\left(\frac{-\left(\tan\left(\frac{l\phi}{4}\right)\right)}{\sigma}\right)\right]^{-2}\exp\left(-\left(\frac{\tan\left(\frac{l\phi}{4}\right)}{\sigma}\right)\right),$$

$$0<\phi<\frac{2\pi}{l},\ \sigma>0 \text{ and } l=1,2,.. \qquad (2.4.1)$$

We call it as **Stereographic -l-axial Logistic distribution**

Case (1) When $l=1$, in the probability density function (5.1), we get the density function

$$g(\phi)=\frac{1}{2\sigma}\sec^2\left(\frac{\phi}{4}\right)\left[1+\exp\left(\frac{-\left(\tan\left(\frac{\phi}{4}\right)\right)}{\sigma}\right)\right]^{-2}\exp\left(-\left(\frac{\tan\left(\frac{\phi}{4}\right)}{\sigma}\right)\right),$$

$$0 < \theta < 2\pi \text{ and } \sigma > 0 \quad (2.4.2)$$

We call it as **Stereographic Circular Logistic distribution**.

Case (2) When $l = 2$, the probability density function (5.1) is the same as that of **Stereographic Semicircular Half Logistic Distribution**

CHAPTER 3

Stereographic Semicircular Quasi Lindley distribution

3.1 Stereographic Semicircular Quasi Lindley Distribution

Definition: A random variable X on the real line is said to have Quasi Lindley Distribution with scale parameter $\alpha > 0$, shape parameter $\sigma > 0$ and location parameter α if the probability density and cumulative distribution functions of X are respectively given by

$$f(x;\alpha,\sigma)=\frac{\alpha}{(1+\sigma)}(\sigma+\alpha x)\exp(-\alpha x),\ \sigma>-1,\ 0<\alpha \text{ and } 0<x<\infty \tag{3.1.1}$$

and

$$F(x;\alpha,\sigma)=1-\exp(-\alpha x)\left[1+\frac{\alpha x}{(\sigma+1)}\right],\ \sigma>-1,\ 0<\alpha \text{ and } 0<x<\infty \tag{3.1.2}$$

Then by applying modified inverse Stereographic projection defined by a one to one mapping $x=v\tan\left(\frac{\theta}{2}\right), v\in\mathbb{R}^{+}$, which leads to a Semicircular Model on unit semicircle. We call this model as Stereographic Semicircular Quasi Lindley Distribution.

Definition:

A random variable X_{SC} on the Semicircle is said to have the Stereographic Semicircular Quasi Lindley distribution with shape parameter $\sigma > -1$, location parameter μ and scale parameter $\sigma > 0$ denoted by **SSCQLD**(σ,λ,μ), if the probability density and the cumulative distribution functions are respectively given by

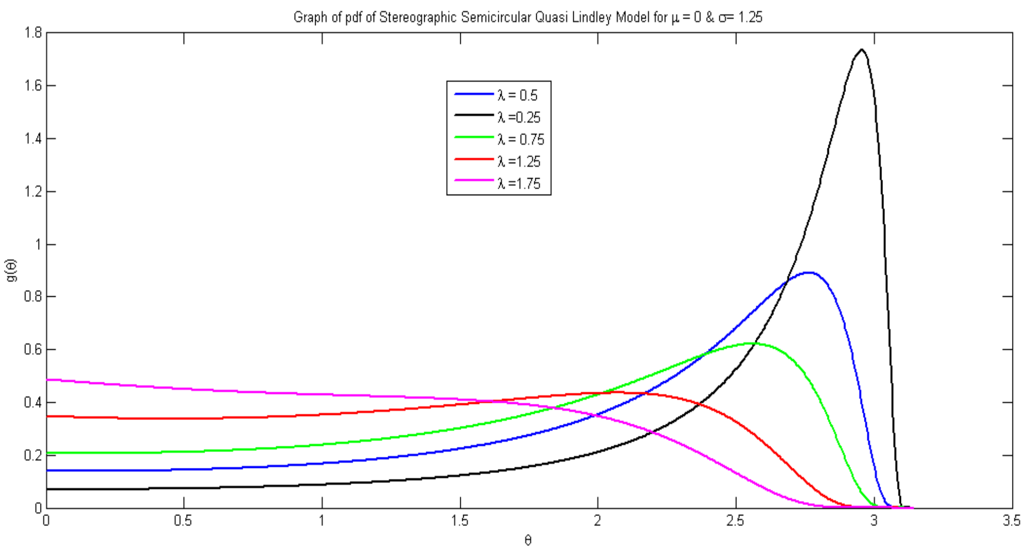
$$g(\theta)=\frac{\lambda\sec^2\left(\frac{\theta-\mu}{2}\right)}{2(\sigma+1)}\left(\sigma+\lambda\tan\left(\frac{\theta-\mu}{2}\right)\right)\exp\left(-\lambda\tan\left(\frac{\theta-\mu}{2}\right)\right),$$

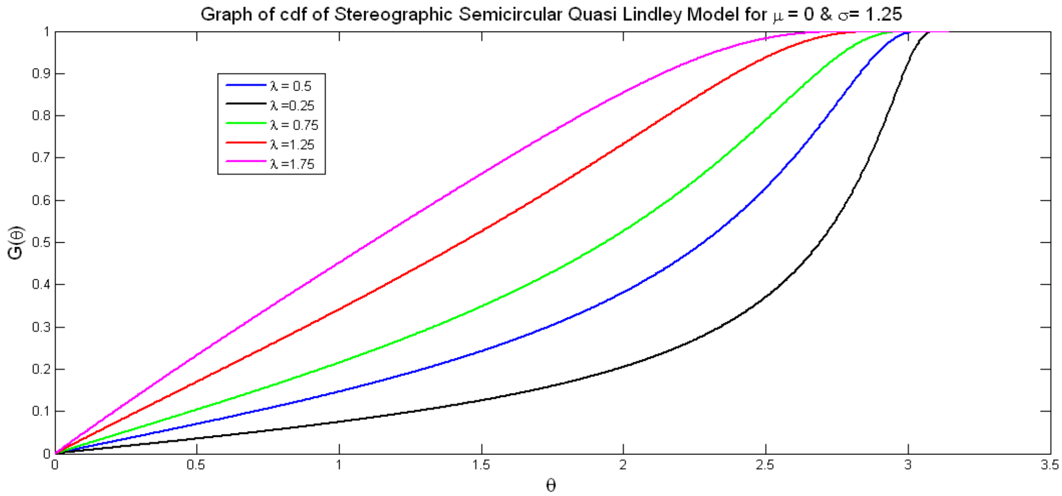
Where $0\le\mu,\theta<\pi,\ \lambda=v\alpha>0\,\text{and}\,\sigma>-1$ (3.1.3)

$$G(\theta)=1-\exp\left(-\lambda\tan\left(\frac{\theta-\mu}{2}\right)\right)\left[1+\frac{\lambda\tan\left(\frac{\theta-\mu}{2}\right)}{(\sigma+1)}\right]$$ Where

$0\le\mu,\theta<\pi,\ \lambda=v\alpha>0\,\text{and}\,\sigma>-1$ (3.1.4)

Graphs of pdf and cdf of Stereographic Semicircular Quasi Lindley Distribution for various values of parameters





3.2 **Trigonometric moments of Stereographic Semicircular Quasi Lindley Distribution**

It is customary to derive the trigonometric moments when a new distribution is proposed. Without loss of generality here we assume that $\mu = 0,$ in (3.3) .The trigonometric moments of the distribution are given by $\{\varphi_p : p = 0, \pm 1, \pm 2, \pm 3, ...\}$, where $\varphi_p = \alpha_p + i\beta_p,$ with $\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta)$ being the p^{th} order cosine and sine moments of the random angle θ, respectively.

Theorem 3.1 Under the pdf of Stereographic Semicircular Quasi Lindley Distribution with $\mu = 0,$ the first four $\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta),$ $p = 1, 2, 3, 4,$ are given as follows:

$$\alpha_1 = 1 - \frac{\lambda\sigma}{\sqrt{\pi}(\sigma+1)} G_{13}^{31}\left(\frac{\lambda^2}{4}\left| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix} \right.\right) - \frac{\lambda^2}{\sqrt{\pi}(\sigma+1)} G_{13}^{31}\left(\frac{\lambda^2}{4}\left| \begin{matrix} -1 \\ -1, 0, \frac{1}{2} \end{matrix} \right.\right),$$

$$\beta_1 = \frac{\lambda\sigma}{\sqrt{\pi}(\sigma+1)} G_{13}^{31}\left(\frac{\lambda^2}{4}\left| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix} \right.\right) + \frac{\lambda^2}{\sqrt{\pi}(\sigma+1)} G_{13}^{31}\left(\frac{\lambda^2}{4}\left| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix} \right.\right),$$

$$\alpha_2 = 1 + \frac{4\lambda\sigma}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} -\frac{3}{2} \\ -\frac{1}{2},0,\frac{1}{2} \end{matrix} \right) + G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2},0,\frac{1}{2} \end{matrix} \right) \right]$$

$$-\frac{4\lambda^2}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2},0,\frac{1}{2} \end{matrix} \right) + G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} -1 \\ -1,0,\frac{1}{2} \end{matrix} \right) \right]$$

$$\beta_2 = \frac{2\lambda\sigma}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} 0 \\ 0,0,\frac{1}{2} \end{matrix} \right) - 2G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} -1 \\ 0,0,\frac{1}{2} \end{matrix} \right) \right]$$

$$+\frac{2\lambda^2}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2},0,\frac{1}{2} \end{matrix} \right) - 2G_{13}^{31}\left(\frac{\lambda^2}{4} \middle| \begin{matrix} -\frac{3}{2} \\ -\frac{1}{2},0,\frac{1}{2} \end{matrix} \right) \right]$$

Proof:

$$\varphi_{\text{p}} = \int_0^{\pi} e^{ip\theta} g(\theta)\,d\theta = \int_0^{\pi} \cos(\text{p}\theta)\, g(\theta)\,d\theta + i\int_0^{\pi} \sin(\text{p}\theta)\, g(\theta)\,d\theta$$

$$= E(\cos(\text{p}\theta)) + i\,E(\sin(\text{p}\theta)) = \alpha_{\text{p}} + i\,\beta_{\text{p}} \quad \text{for } \text{p} = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

For the first cosine and sine moments, use the transformations $x = \tan\left(\frac{\theta}{2}\right)$, $\cos\theta = 1 - \frac{2x^2}{1+x^2}$ and $\sin\theta = \frac{2x}{1+x^2}$,the results α_1 and β_1 follows by the integral formula **3.389.2** (Gradshteyn and Ryzhik, 2007).

Now $\text{E}(\cos(\text{p}\theta)) = \frac{\lambda}{2(\sigma+1)}\int_0^{\pi} \cos(\text{p}\theta)\sec^2\left(\frac{\theta}{2}\right)\left(\sigma + \lambda\tan\left(\frac{\theta}{2}\right)\right) e^{-\lambda\left(\tan\left(\frac{\theta}{2}\right)\right)} d\theta$ and

$$\mathrm{E}\left(\sin\left(\mathrm{p}\theta\right)\right)=\frac{\lambda}{2\left(\sigma+1\right)}\int_{0}^{\pi}\sin\left(\mathrm{p}\theta\right)\sec^{2}\left(\frac{\theta}{2}\right)\left(\sigma+\lambda\tan\left(\frac{\theta}{2}\right)\right)e^{-\lambda\left(\tan\left(\frac{\theta}{2}\right)\right)}d\theta$$

$$\alpha_{1}=\frac{\lambda}{2\left(\sigma+1\right)}\int_{0}^{\pi}\cos\theta\sec^{2}\left(\frac{\theta}{2}\right)\left(\sigma+\lambda\tan\left(\frac{\theta}{2}\right)\right)e^{-\lambda\left(\tan\left(\frac{\theta}{2}\right)\right)}d\theta$$

$$=\frac{\lambda}{\left(\sigma+1\right)}\int_{0}^{\infty}\left[1-\frac{2x^{2}}{1+x^{2}}\right]\left(\sigma+\lambda x\right)e^{-\lambda x}dx$$

$$=1-\frac{2\lambda\sigma}{\left(\sigma+1\right)}\int_{0}^{\infty}\frac{x^{2}}{1+x^{2}}e^{-\lambda x}dx-\frac{2\lambda^{2}}{\left(\sigma+1\right)}\int_{0}^{\infty}\frac{x^{3}}{1+x^{2}}e^{-\lambda x}dx$$

$$=1-\frac{2\lambda\sigma}{\left(\sigma+1\right)}\int_{0}^{\infty}x^{2\left(\frac{3}{2}\right)-1}\left(1+x^{2}\right)^{0-1}e^{-\lambda x}dx-\frac{2\lambda^{2}}{\left(\sigma+1\right)}\int_{0}^{\infty}x^{2(2)-1}\left(1+x^{2}\right)^{0-1}e^{-\lambda x}dx$$

$$=1-\frac{2\lambda\sigma}{\left(\sigma+1\right)}\left[\frac{1}{2\sqrt{\pi}}G_{13}^{31}\left(\frac{\lambda^{2}}{4}\middle|\begin{matrix}-\frac{1}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)\right]-\frac{2\lambda^{2}}{\left(\sigma+1\right)}\left[\frac{1}{2\sqrt{\pi}}G_{13}^{31}\left(\frac{\lambda^{2}}{4}\middle|\begin{matrix}-1\\-1,0,\frac{1}{2}\end{matrix}\right)\right]$$

$$\alpha_{1}=1-\frac{\lambda\sigma}{\sqrt{\pi}\left(\sigma+1\right)}G_{13}^{31}\left(\frac{\lambda^{2}}{4}\middle|\begin{matrix}-\frac{1}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)-\frac{\lambda^{2}}{\sqrt{\pi}\left(\sigma+1\right)}G_{13}^{31}\left(\frac{\lambda^{2}}{4}\middle|\begin{matrix}-1\\-1,0,\frac{1}{2}\end{matrix}\right)$$

$$\beta_{1}=\frac{\lambda}{2\left(\sigma+1\right)}\int_{0}^{\pi}\sin\theta\sec^{2}\left(\frac{\theta}{2}\right)\left(\sigma+\lambda\tan\left(\frac{\theta}{2}\right)\right)e^{-\lambda\left(\tan\left(\frac{\theta}{2}\right)\right)}d\theta$$

$$=\frac{2\lambda}{\left(\sigma+1\right)}\int_{0}^{\infty}\left(\frac{x}{1+x^{2}}\right)\left(\sigma+\lambda x\right)e^{-\lambda x}dx$$

$$=\frac{2\lambda\sigma}{\left(\sigma+1\right)}\int_{0}^{\infty}x^{2(1)-1}\left(1+x^{2}\right)^{0-1}e^{-\lambda x}dx+\frac{2\lambda^{2}}{\left(\sigma+1\right)}\int_{0}^{\infty}x^{2\left(\frac{3}{2}\right)-1}\left(1+x^{2}\right)^{0-1}e^{-\lambda x}dx$$

$$\beta_1 = \frac{\lambda\sigma}{\sqrt{\pi}(\sigma+1)} G\left(\frac{\lambda^2}{4}\middle|\begin{matrix}0\\0,0,\frac{1}{2}\end{matrix}\right) + \frac{\lambda^2}{\sqrt{\pi}(\sigma+1)} G\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-\frac{1}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)$$

To obtain second cosine and sine moments α_2 and β_2, we use the transformations $x = \tan\left(\frac{\theta}{2}\right)$, $\cos 2\theta = 1 + \frac{8x^4}{\left(1+x^2\right)^2} - \frac{8x^2}{\left(1+x^2\right)}$ and $\sin 2\theta = \frac{4x}{\left(1+x^2\right)} - \frac{8x^3}{\left(1+x^2\right)^2}$, the results of α_2 and β_2 follows by the same integral formula of α_1.

$$\alpha_2 = \frac{\lambda}{2(\sigma+1)} \int_0^{\pi} \cos 2\theta \sec^2\left(\frac{\theta}{2}\right)\left(\sigma + \lambda\tan\left(\frac{\theta}{2}\right)\right) e^{-\lambda\left(\tan\left(\frac{\theta}{2}\right)\right)} d\theta$$

$$= \frac{\lambda}{(\sigma+1)} \int_0^{\infty} \left[1 + \frac{8x^4}{\left(1+x^2\right)^2} - \frac{8x^2}{\left(1+x^2\right)}\right](\sigma+\lambda x) e^{-\lambda x} dx$$

$$= 1 + \frac{8\lambda}{(\sigma+1)} \int_0^{\infty} \frac{x^4}{\left(1+x^2\right)^2}(\sigma+\lambda x) e^{-\lambda x} dx - \frac{8\lambda}{(\sigma+1)} \int_0^{\infty} \frac{x^2}{\left(1+x^2\right)}(\sigma+\lambda x) e^{-\lambda x} dx$$

$$= 1 + \frac{8\lambda\sigma}{(\sigma+1)} \int_0^{\infty} x^{2\left(\frac{5}{2}\right)-1}\left(1+x^2\right)^{-1-1} e^{-\lambda x} dx + \frac{8\lambda^2}{(\sigma+1)} \int_0^{\infty} x^{2(3)-1}\left(1+x^2\right)^{-1-1} e^{-\lambda x} dx$$

$$- \frac{8\lambda\sigma}{(\sigma+1)} \int_0^{\infty} x^{2\left(\frac{3}{2}\right)-1}\left(1+x^2\right)^{0-1} e^{-\lambda x} dx - \frac{8\lambda^2}{(\sigma+1)} \int_0^{\infty} x^{2(2)-1}\left(1+x^2\right)^{0-1} e^{-\lambda x} dx$$

$$\alpha_2 = 1 + \frac{4\lambda\sigma}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-\frac{3}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right) + G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-\frac{1}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)\right]$$

$$-\frac{4\lambda^2}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-\frac{1}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)+G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-1\\-1,0,\frac{1}{2}\end{matrix}\right)\right]$$

$$\beta_2=\frac{\lambda}{2(\sigma+1)}\int_0^{\pi}\sin 2\theta\sec^2\left(\frac{\theta}{2}\right)\left(\sigma+\lambda\tan\left(\frac{\theta}{2}\right)\right)e^{-\lambda\left(\tan\left(\frac{\theta}{2}\right)\right)}d\theta$$

$$=\frac{\lambda}{(\sigma+1)}\int_0^{\infty}\left[\frac{4x}{(1+x^2)}-\frac{8x^3}{(1+x^2)^2}\right](\sigma+\lambda x)e^{-\lambda x}dx$$

$$=\frac{\lambda}{(\sigma+1)}\int_0^{\infty}\frac{4x}{(1+x^2)}(\sigma+\lambda x)e^{-\lambda x}dx-\frac{\lambda}{(\sigma+1)}\int_0^{\infty}\frac{8x^3}{(1+x^2)^2}(\sigma+\lambda x)e^{-\lambda x}dx$$

$$=\frac{4\lambda\sigma}{(\sigma+1)}\int_0^{\infty}x^{2(1)-1}(1+x^2)^{0-1}e^{-\lambda x}dx+\frac{4\lambda\sigma}{(\sigma+1)}\int_0^{\infty}x^{2\left(\frac{3}{2}\right)-1}(1+x^2)^{0-1}e^{-\lambda x}dx$$

$$-\frac{8\lambda\sigma}{(\sigma+1)}\int_0^{\infty}x^{2(2)-1}(1+x^2)^{-1-1}e^{-\lambda x}dx-\frac{8\lambda^2}{(\sigma+1)}\int_0^{\infty}x^{2\left(\frac{5}{2}\right)-1}(1+x^2)^{-1-1}e^{-\lambda x}dx$$

$$\beta_2=\frac{2\lambda\sigma}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}0\\0,0,\frac{1}{2}\end{matrix}\right)-2G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-1\\0,0,\frac{1}{2}\end{matrix}\right)\right]$$

$$+\frac{2\lambda^2}{\sqrt{\pi}(\sigma+1)}\left[G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-\frac{1}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)-2G_{13}^{31}\left(\frac{\lambda^2}{4}\middle|\begin{matrix}-\frac{3}{2}\\-\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)\right]$$

Higher-order moments can be obtained similarly.

The first two trigonometric moments are sufficient for calculating population characteristics .

3.3 Stereographic -l-axial Quasi Lindley distribution

We extend the above Stereographic Semicircular model to the l -axial distribution, which is applicable to any arc of arbitrary length say $2\pi/l$ for $l=1,2,...,$ so it is desirable to extend the Stereographic Semicircular Quasi Lindley distribution to construct the Stereographic- l -axial Quasi Lindley distribution, we consider the density function of Stereographic Semicircular Quasi Lindley distribution and use the transformation $\phi = 2\theta/l$, $l=1,2,...,$. The probability density function of ϕ is given by

$$g(\theta) = \frac{\lambda l \sec^2\left(\frac{l\theta}{4}\right)}{4(\sigma+1)}\left(\sigma + \lambda \tan\left(\frac{l\theta}{4}\right)\right)\exp\left(-\lambda \tan\left(\frac{l\theta}{4}\right)\right),$$

$$0 < \theta < \frac{2\pi}{l},\ \sigma > -1, \lambda > 0 \text{ and } l = 1,2,.. \quad (3.3.1)$$

We call it as **Stereographic -l-axial Quasi Lindley distribution.**

Case (1) When $l=1$, in the probability density function (4.1), we get the density function

$$g(\theta) = \frac{\lambda \sec^2\left(\frac{\theta}{4}\right)}{4(\sigma+1)}\left(\sigma + \lambda \tan\left(\frac{\theta}{4}\right)\right)\exp\left(-\lambda \tan\left(\frac{\theta}{4}\right)\right),$$

$$0 < \theta < 2\pi,\ \sigma > -1, \lambda > 0 \quad (3.3.2)$$

We call it as **Stereographic Circular Quasi Lindley distribution**.

Case (2) When $l=2$, the probability density function (4.1) is the same as that of **Stereographic Semicircular Quasi Lindley Distribution.**

CHAPTER 4

Stereographic Semicircular Erlang distribution

4.1 Stereographic Semicircular Erlang Distribution

The Erlang distribution was introduced by Agner.K. Erlang is special case of the gamma distribution, where the shape parameter is positive integer. It is a continuous probability distribution with support on $(0,\infty)$, and has wide range of applications in fields like traffic engineering , stochastic processes and biomathematics, mainly due to its relative to the exponential distribution.

Here we recall the definition of Erlang distribution.

Definition 4.1 A continuous random variable X is said to follow Erlang distribution with shape parameter n (a positive integer), location parameter η and scale parameter $\lambda > 0$, if its probability density and distribution functions are respectively given by

$$f(x) = \frac{(x-\eta)^{n-1}}{\lambda^n \Gamma(n)} e^{-\left(\frac{x-\eta}{\lambda}\right)}, \text{ where } n \in Z^+, \lambda > 0, \eta < x < \infty. \tag{4.1.1}$$

$$F(x) = \frac{\gamma(n, \lambda x)}{(n-1)!}, \text{ where } \gamma(.) \text{ is the lower incomplete gamma function} \tag{4.1.2}$$

Definition 4.2 A random variable θ_{SC} on the semicircle is said to have the Stereographic semicircular Erlang distribution with shape parameter n ,(n is a positive integer), scale parameter $\sigma > 0$ and location parameter μ, denoted by **SSCED** (n, σ, μ), if the probability density and the distribution functions are respectively given by

$$g(\theta)=\frac{\sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)^{n-1}}{2\sigma^n\Gamma(n)}\exp\left(-\left(\frac{\tan\left(\frac{\theta}{2}\right)-\mu}{\sigma}\right)\right) \qquad (4.1.3)$$

Where $n\in Z^+, \sigma=\frac{\lambda}{\upsilon}>0, \mu=\frac{\eta}{\upsilon}, 0<\theta<\pi.$

$$G(\theta)=\frac{\gamma\left(n,\sigma\tan\left(\frac{\theta}{2}\right)\right)}{(n-1)!}, \qquad (4.1.4)$$

where $\gamma(.)$ is the lower incomplete gamma function.

If $n=1$, the Stereographic semicircular Erlang distribution reduces to Stereographic semicircular exponential distribution (Phani et al.(2013)).

Graphs of probability density function of Stereographic Semicircular Erlang Distribution

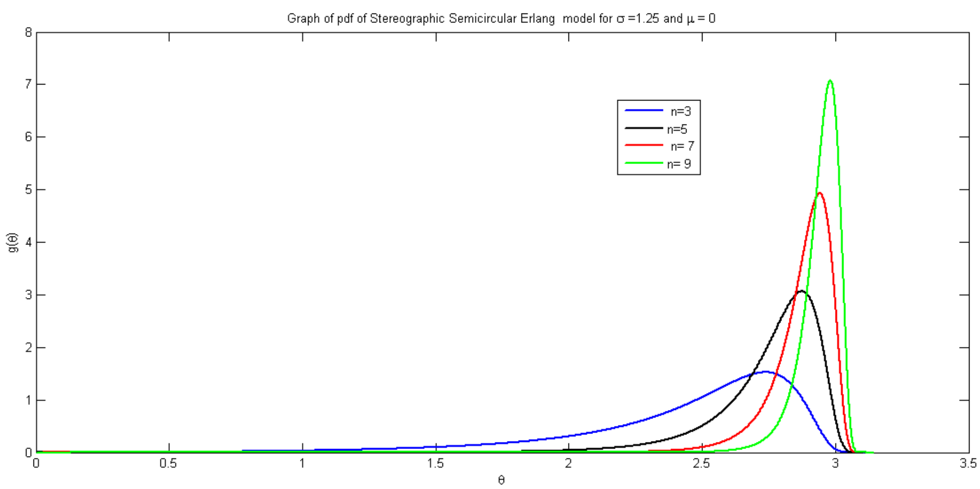


Fig.1

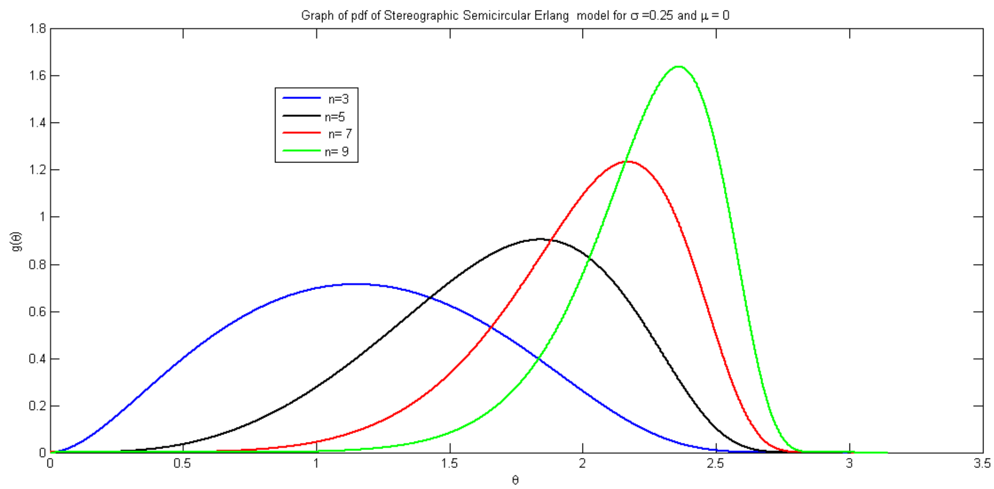


Fig.2

4.2 Trigonometric Moments

The trigonometric moments of the distribution are given by $\{\varphi_p : p = 0, \pm 1, \pm 2, \pm 3, \ldots\}$, where with $\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta)$ being the p^{th} order cosine and sine moments of the random angle θ, respectively. Here we derive the first two trigonometric moments for the proposed model, which plays a prominent role in computing population characteristics.

Theorem 4.1 Under the pdf of Stereographic Semicircular Erlang Distribution with $\mu = 0$, the first two $\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta)$, $p = 1, 2.$ are given as follows:

$$\alpha_1 = 1 - \frac{2n}{\sqrt{\pi}\ \sigma^n n!} G_{13}^{31}\left(\frac{1}{4\sigma^2} \middle| \begin{matrix} -\frac{n}{2} \\ -\frac{n}{2}, 0, \frac{1}{2} \end{matrix} \right), \qquad (4.2.1)$$

$$\alpha_2 = 1 - \frac{4n}{\sqrt{\pi}\,\sigma^n\ n!} G_{13}^{31}\left(\frac{1}{4\sigma^2} \middle| \begin{matrix} -\frac{n}{2} \\ \left(\frac{2-n}{2}\right), 0, \frac{1}{2} \end{matrix} \right), \qquad (4.2.2)$$

$$\beta_1 = \frac{n}{\sqrt{\pi}\ \sigma^n n!} G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}\left(\frac{1-n}{2}\right) \\ \left(\frac{1-n}{2}\right), 0, \frac{1}{2}\end{matrix}\right.\right), \tag{4.2.3}$$

$$\beta_2 = \frac{2n}{\sqrt{\pi}\ \sigma^n n!}\left[G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}\left(\frac{1-n}{2}\right) \\ \left(\frac{3-n}{2}\right), 0, \frac{1}{2}\end{matrix}\right.\right) - G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\left(\frac{1+n}{2}\right) \\ \left(\frac{1-n}{2}\right), 0, \frac{1}{2}\end{matrix}\right.\right)\right]. \tag{4.2.4}$$

Proof:

To find the first cosine moment $\alpha_1 = \int_0^{\pi} \cos\theta g(\theta) d\theta$

$$\alpha_1 = \frac{n}{2n!\sigma^n}\int_0^{\pi} \cos\theta \sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{n-1} e^{-\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}} d\theta$$

Using the transformation $x = \tan\left(\frac{\theta}{2}\right)$ and $\cos\theta = 1 - \frac{2x^2}{1+x^2}$, we get

$$\alpha_1 = \frac{n}{n!\sigma^n}\int_0^{\infty}\left(1 - \frac{2x^2}{1+x^2}\right) x^{n-1} e^{-\frac{x}{\sigma}} dx$$

$$= 1 - \frac{2n}{n!\sigma^n}\int_0^{\infty} x^{n+1}\left(1+x^2\right)^{0-1} e^{-\frac{x}{\sigma}} dx$$

$$= 1 - \frac{2n}{n!\sigma^n}\int_0^{\infty} x^{2\left(\frac{n+2}{2}\right)-1}\left(1+x^2\right)^{0-1} e^{-\frac{x}{\sigma}} dx$$, the result α_1 follows by the integral formula **3.389.2 (**Gradshteyn and Ryzhik, 2007).

To find the second cosine moment, $\alpha_2 = \int_0^{\pi} \cos 2\theta g(\theta) d\theta$

$$\alpha_2 = \frac{n}{2n!\sigma^n}\int_0^{\pi} \cos 2\theta \sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{n-1} e^{-\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}} d\theta$$

Using the transformation $x = \tan\left(\frac{\theta}{2}\right)$ and $\cos 2\theta = 1 - \frac{8x^2}{\left(1+x^2\right)^2}$, we get

$$\alpha_2 = \frac{n}{n!\sigma^n}\int_0^\infty \left(1 - \frac{8x^2}{\left(1+x^2\right)^2}\right) x^{n-1} e^{-\frac{x}{\sigma}} dx$$

$$= 1 - \frac{8n}{n!\sigma^n}\int_0^\infty x^{n+1}\left(1+x^2\right)^{-2} e^{-\frac{x}{\sigma}} dx$$

$$= 1 - \frac{8n}{n!\sigma^n}\int_0^\infty x^{2\left(\frac{n+2}{2}\right)-1}\left(1+x^2\right)^{-1-1} e^{-\frac{x}{\sigma}} dx$$, the result of α_2 follows by the same integral formula of α_1.

To find the first sine moment, $\beta_1 = \int_0^\pi \sin\theta g(\theta) d\theta$

$$\beta_1 = \frac{n}{2n!\sigma^n}\int_0^\pi \sin\theta \sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{n-1} e^{-\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}} d\theta$$

Using the transformation $x = \tan\left(\frac{\theta}{2}\right)$ and $\sin\theta = \frac{2x}{1+x^2}$, we get

$$\beta_1 = \frac{n}{n!\sigma^n}\int_0^\infty \frac{2x}{1+x^2} x^{n-1} e^{-\frac{x}{\sigma}} dx$$

$$= \frac{2n}{n!\sigma^n}\int_0^\infty x^n \left(1+x^2\right)^{0-1} e^{-\frac{x}{\sigma}} dx$$

$$= \frac{2n}{n!\sigma^n}\int_0^\infty x^{2\left(\frac{n+1}{2}\right)-1}\left(1+x^2\right)^{0-1} e^{-\frac{x}{\sigma}} dx$$, the result of β_1 follows by the same integral formula of α_1.

To find the second sine moment $\beta_2 = \int_0^\pi \sin 2\theta g(\theta) d\theta$

$$\beta_2 = \frac{n}{2n!\sigma^n}\int_0^{\pi} \sin 2\theta \sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{n-1} e^{-\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}} d\theta$$

Using the transformation $x = \tan\left(\frac{\theta}{2}\right)$ and $\sin 2\theta = \frac{4x}{1+x^2} - \frac{4x^3}{\left(1+x^2\right)^2}$, we get

$$\beta_2 = \frac{n}{n!\sigma^n}\int_0^{\infty}\left[\frac{4x}{1+x^2} - \frac{4x^3}{\left(1+x^2\right)^3}\right] x^{n-1} e^{-\frac{x}{\sigma}} dx$$

$$= \frac{4n}{n!\sigma^n}\left[\int_0^{\infty} x^n \left(1+x^2\right)^{0-1} e^{-\frac{x}{\sigma}} dx - \int_0^{\infty} x^{n+2}\left(1+x^2\right)^{-1-1} e^{-\frac{x}{\sigma}} dx\right]$$

$$= \frac{4n}{n!\sigma^n}\left[\int_0^{\infty} x^{2\left(\frac{n+1}{2}\right)-1}\left(1+x^2\right)^{0-1} e^{-\frac{x}{\sigma}} dx - \int_0^{\infty} x^{2\left(\frac{n+3}{2}\right)-1}\left(1+x^2\right)^{-1-1} e^{-\frac{x}{\sigma}} dx\right],$$ the result of β_2 follows by the same integral formula of α_1.

by similar process, higher-order moments can be obtained.

4.3 Extension to l-axial Model

We extend the proposed model to the $l-$axial distribution, which is applicable to any arc of arbitrary length say $\frac{2\pi}{l}$ for $l = 1, 2, 3, \ldots,$. To construct the $l-$axial semicircular Erlang distribution, we consider the density function of proposed model and use the transformation $\phi = \frac{2\theta}{l}$, $l = 1, 2, 3, \ldots,$. The probability density function of ϕ is given by

$$g(\phi) = \frac{l \sec^2\left(\frac{l\phi}{4}\right)\left(\tan\left(\frac{l\phi}{4}\right)\right)^{n-1}}{4\sigma^n \Gamma(n)} \exp\left(-\left(\frac{\tan\left(\frac{l\phi}{4}\right)}{\sigma}\right)\right) \quad (4.3.1)$$

Where $n \in Z^+, \sigma = \frac{\lambda}{\upsilon} > 0$ and $0 < \phi < \frac{2\pi}{l}$.

Special cases

Case (1) When $l = 2$, the probability density function (4.3.1) is the same as the probability density function of Stereographic semicircular Erlang distribution.

Case (2) When $l = 1$, the probability density function (4.3.1) is

$$g(\phi) = \frac{\sec^2\left(\frac{\phi}{4}\right)\left(\tan\left(\frac{\phi}{4}\right)\right)^{n-1}}{4\sigma^n \Gamma(n)} \exp\left(-\left(\frac{\tan\left(\frac{\phi}{4}\right)}{\sigma}\right)\right) \tag{4.3.2}$$

Where $n \in Z^+, \sigma = \frac{\lambda}{\upsilon} > 0$ and $0 < \phi < 2\pi$. which is a circular distribution, we call it as Stereographic Circular Erlang Distribution.

Case (3) When $l = 2$ and $n = 1$, the probability density function (4.3.1) is

$$g(\phi) = \frac{\sec^2\left(\frac{\phi}{2}\right)}{2\sigma} \exp\left(-\left(\frac{\tan\left(\frac{\phi}{2}\right)}{\sigma}\right)\right) \tag{4.3.3}$$

Where $\sigma > 0$ and $0 < \phi < \pi.$, which is the probability density function of Stereographic semicircular exponential distribution(Phani et al.(2013)).

References

1. Abramowitz, M.& Stegun, I.A. (1965), Handbook of Mathematical Functions, *Dover, New York*.
2. Bakouch HS, Al-Zahrani BM, Al-Shomrani AA, Marchi VAA, Louzada F (2012). An extended Lindley distribution. J. Korean Stat. Soc. 41(1):pp. 75-85.
3. Byoung Jin Ahn and Hyoung – Moon Kim (2008), A New Family f Semicircular Models: The Semicircular Laplace Distributions, Communications of the Korean Statistical Society, Vol. 15, No. 5, pp.775 – 781.
4. Dattatreya Rao, A.V., Ramabhadra Sarma, I. and Girija, S.V.S. (2007), On Wrapped Version of Some Life Testing Models, *Comm Statist,- Theor.Meth*. , 36, issue # 11, pp.2027-2035.
5. Dattatreya Rao., Girija S.V.S. and Phani (2014). Stereographic Logistic Model – Application to Noisy Scrub Birds Data, Chilean Journal of Statistics (to be appeared).
6. Fisher, N. I. (1993). Statistical Analysis of Circular Data. Cambridge University Press, Cambridge.
7. Girija S.V.S., A.V.Dattatreya Rao and Phani Yedlapalli(2014) New Circular model induced by Inverse Stereographic projection on Double Exponential Model-Application to Birds Migration Data. Journal of Applied Mathematics, Statistics and Informatics (JAMSI). Vol 10, No. 1, pp.5-17.
8. Gradshteyn and Ryzhik (2007). Table of Integrals, series and products, 7th edition, Academic Press.
9. Guardiola, J.H. (2004). The Semicircular Normal Distribution, Ph.D Dissertation, Baylor University, Institute of Statistics.
10. Hyoung-Moon Kim, (2010), A Projected Exponential Family for Modelling Semicircular Data, The Korean Journal of Applied Statistics, Vol.26, No.6, pp.1125-1145.
11. Jammalamadaka S. Rao and Sen Gupta, A. (2001), Topics in Circular Statistics, *World Scientific Press, Singapore.*
12. Minh, Do Le and Farnum, Nicholas R. (2003), Using Bilinear Transformations to Induce Probability Distributions, *Communication in Statistics – Theory and Methods*, 32, 1, pp. 1 – 9.
13. Mardia, K.V. and Jupp, P.E. (2000), Directional Statistics, *John Wiley, Chichester.*
14. Olapade, A. K. (2003), On Characterizations of the Half Logistic Distribution. Inter Stat, February Issue,**2**, http://interstat.stat.vt.edu/InterStat/ARTICLES/2003articles/F06002.

15. Phani.Y, Girija S.V.S., and Dattatreya Rao A.V. (2013). On Construction of Stereographic Semicircular Models, Journal of Applied Probability and Statistics. Vol 8, No. 1, pp. 75-90.

16. Rama S and Mishra A (2013). A quasi Lindley distribution. African Journal of Mathematics and Computer Science Research.6(4),pp. 64-71

17. Toshihiro Abe, Kunio Shimizu and Arthur Pewsey, (2010), Symmetric Unimodal Models for Directional Data Motivated by Inverse Stereographic Projection, *J. Japan Statist. Soc.,* Vol. 40 (No. 1), pp. 45-61.

18. Ugai. S.K., Nishijima, M. and Kan, T. (1977), Characteristics of raindrop size and raindrop shape, *Open symposium* URSI *Commission F*, 225-230.