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on

ON TRIDIAGONAL MATRICES

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DECLARATION

We hereby declare that the **Management, Hindu College, Guntur** sponsored Minor Research Project report titled **ON TRIDIAGONAL MATRICES** comprises of our own and original work. It has not been submitted fully or partially to any other institution or organization and is not published.

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CERTIFICATE

Certified that this is a genuine and bonafide work done by **Y. SREEKANTH**, Lecturer in Mathematics with the Minor Research Project along with Dr. S.V.S. GIRIJA, Associate Professor of Mathematics titled **ON TRIDIAGONAL MATRICES** sanctioned by **Management, Hindu College, Guntur**.

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CONTENTS

INTRODUCTION

Matrices in general, and tridiagonal matrices inparticular, play an important role in several areas of Applied Sciences. We encounter tridiagonal matrices very often in Multivariate analysis , Image analysis, Financial market studies, Circular data analysis, Biomathematics, to mention a few. In "Parallel Computing " there is a mention about cholesky decomposition of tridiagonal symmetric matries [Ilan Bar On in 1995] . There is also a mention about tridiagonal matrices in Mixture analysis in Fisheries and Distributed large – scale visualization in Networking problems. It is well known that there exists an analytic expression for the eigenvalues of a tridiagonal matrix of the type

$$\begin{pmatrix} a & b & & & & & \\ c & a & b & & & & \\ & c & a & b & & & \\ & & .. & .. & .. & & \\ & & & .. & .. & .. & \\ & & & & c & a & b \\ & & & & & c & a \end{pmatrix}$$

In early 1990's it was a hot problem to extend the result to a block – tridiagonal matrix required in certain algorithms of computer networks.The wide range of applications and various algorithms for matrix operations on tridiagonal matrices , we are motivated to make a detailed study of this topic and present our observations and our own results along with algorithms on tridiagonal matrices and relatives in the form of this dissertation. We observe the properties of the set of tridiagonal matrices and the set of the lower tridiagonal matrices and adopt some techniques developed by Rami Reddy, B., [7] and apply to the tridiagonal matrices for finding inverse of nonsingular tridiagonal matrices and lower tridiagonal matrices by recursion method. These form Section 1 of this dissertation.

LU decomposition for matrix inversion is a basic problem in numerical analysis where the complexity studies have yielded fruitful results. LU decomposition is also suited to compute eigenvalues and eigenvectors of matrices. The concept of LU decomposition is taken from [2] and arrive at new algorithm for finding LU decomposition of tridiagonal matrix and symmetric , positive definite and tridiagonal matrix by single bordering in Section 2.

In the progress of work , we come across the special type of symmetric, positive definite and tridiagonal matrices in which (2k, 2k-1) entries are zero $\forall$ k.These matrices possess some special features and are discussed in Section – 3. We present a few results and algorithms of our own obtained in this process. These results will avoid many a hurdle in the adoption of tridiagonal matrices in various streams . Number of computations will also be reduced in finding inverse of a matrix by cholesky decomposition.

In the process of survey for tridiagonal matrices we find a beautiful and very handy method [1] . In this method an unsymmetrical tridiagonal matrix shall be transformed to symmetrical form which simplifies the computation of eigenvalues of tridiagonal matrices. Breaking down method [1] for symmetric tridiagonal matrices turns our work easy to find eigenvalues of tridiagonal matrix.
Application of the above method to the special type of symmetric, positive definite and tridiagonal matrices in which (2k, 2k-1) entries are zero $\forall$ k, is discussed in Section – 4.

The concept of QR factorization of a matrix is taken from [4]. In Section – 5 , we present some observations on QR factorization of tridiagonal matrices and special type of tridiagonal matrices in which (2k, 2k-1) entries are zero $\forall$ k.

CHAPTER – I

All the matrices under consideration are square matrices over the field of real numbers. We call the first diagonal below the principal diagonal of a square matrix A, the subdiagonal of A and the diagonal above the principal diagonal as superdiagonal.

A tridiagonal (also called Jacobi) matrix is a matrix which has zeros except on the principal diagonal, the superdiagonal and the subdiagonal. Thus A = (a_{ij}) is tridiagonal when a_{ij} = 0 for |i-j| > 1.

A matrix which has zeros except on the principal diagonal and the subdiagonal is known as **lower tridiagonal matrix** (l t d).

A matrix which has zeros except on the principal diagonal and the superdiagonal is known as **upper tridiagonal matrix** (u t d).

A matrix of the form P = I –2 W W^t where W is a column vector, W$\in R^n$, W^t=(w_1…..w_n) such that W^tW = w_1^2 + w_2^2 + w_3^2 +……w_n^2 =1 is known as **Householder matrix**. Clearly P is symmetric and orthogonal.

An upper triangular matrix which has zeros except on the subdiagonal is known as **upper Hessenberg matrix**.

A = (a_{ij}) is upper Hessenberg matrix when a_{ij} = 0 for I > j+1.

1.1. Proposition : The set TD of all tridiagonal matrices of order n is a vector space of dimension 3n-2.

Proof: Since 0$\in$TD, TD $\neq \phi$. Let A,B $\in$TD.

If A = (a_{ij}) and B = (b_{ij}) then A+B = $(a_{ij} + b_{ij})$

Since a_{i_j} = 0 and b_{i_j} = 0 if | i-j | > 1,

$a_{ij} + b_{ij}$ = 0 if | i-j | > 1

$\Rightarrow$ A + B $\in$ T D.

Hence TD is closed under addition.

Let k $\in$ R. Then kA = k(a_{ij}) = (ka_{ij})
Since a_{ij} = 0 if | i-j | > 1, ka_{ij} = 0 if | i-j | > 1

TD is closed under scalar multiplication. Hence TD is a subspace of the vector space of all matrices of order n and hence TD is a vector space by itself. Write E_{ij} for the matrix of order n, with 1 in the (i, j)th place and zero elsewhere.

Let B = { E_{ij} / | i-j | >1, 1 $\leq$ i, j $\leq$ n }

B $\subseteq$ TD If $\sum_{i,j} x_{ij}E_{ij} = 0,$ then 0 = x_{ij} = $\sum_{i,j} x_{ij}E_{ij} \Rightarrow x_{ij} = 0$

Hence B is linearly independent.

Let A $\in$ TD, A = (a_{ij}) where $a_{ij} = 0$, if | i – j | > 1.

Then A = $\sum_{i,j=1}^{n} A_{ij}$ where A_{ij} has a_{ij} in (i , j)th place and 0 elsewhere

= $\sum_{i,j=1}^{n} a_{ij}E_{ij}$ and $a_{ij} = 0$ if | i – j | > 1.

$\Rightarrow$ B spans TD.

Since B is a linearly independent subset of TD and spans TD, B is a basis for TD.

Since B has (3n-2) elements, dimension TD = 3n-2.

1.2. Proposition: The set LTD of all lower tridiagonal matrices of order n is a subspace of TD and has dimension 2n-1.

Proof: clearly 0 $\in$ LTD $\subseteq$ TD

If A, B $\in$ L T D,
A = (a_{ij}) and B = (b_{ij})
Where $a_{ij} = b_{ij} = 0$ if j – i >1

If x, y $\in$ R then xA + yB = (xa_{ij} +yb_{ij}) = 0 ,if xa_{ij} + yb_{ij} = 0 , for j – i >1

Since $a_{ij} = b_{ij} = 0$ if j – i >1

hence xA + yB $\in$ L T D

Since L T D is a nonempty set and is closed under addition and scalar multiplication, L T D is a subspace of T D.

Let $B_1 = \{ E_{ij}$ / j – i >1, 1 $\leq$ i, j $\leq$ n }

Then $B_1 \subseteq$ T D and B_1 is linearly independent

Let A $\in$ LTD, A = (a_{ij}) where a_{ij} = 0 if j - i >1

Then A = $\sum_{i,j=1}^{n} a_{ij} E_{ij}$ when j – i >1

$\therefore B_1$ spans LTD

Since B_1 is a linearly independent subset of LTD and spans LTD, B_1 is a basis for LTD and dimension LTD = 2n-1.

Remark 1.3: TD and LTD are not closed under multiplication.

Illustration (i)

$$A = \begin{pmatrix} 2 & -4 & 0 & 0 & 0 \\ -1 & 3 & 5 & 0 & 0 \\ 0 & 2 & 4 & 1 & 0 \\ 0 & 0 & 7 & 5 & 6 \\ 0 & 0 & 0 & -3 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 9 & -2 & 0 & 0 & 0 \\ 4 & 7 & 1 & 0 & 0 \\ 0 & 2 & 6 & 8 & 0 \\ 0 & 0 & -9 & 4 & -1 \\ 0 & 0 & 0 & 5 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & -32 & -4 & 0 & 0 \\ 3 & 33 & 33 & 40 & 0 \\ 8 & 22 & 17 & 36 & -1 \\ 0 & 14 & -3 & 106 & 7 \\ 0 & 0 & 27 & -7 & 5 \end{pmatrix}$$

$\therefore AB \notin$ TD.

Illustration (ii):

If

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 7 & 5 & 0 \\ 0 & 0 & 0 & -3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 9 & 0 & 0 & 0 & 0 \\ 4 & 7 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & -9 & 4 & 0 \\ 0 & 0 & 0 & 5 & 2 \end{pmatrix}$$

then

$$CD = \begin{pmatrix} 18 & 0 & 0 & 0 & 0 \\ 3 & 21 & 0 & 0 & 0 \\ 8 & 22 & 24 & 0 & 0 \\ 0 & 14 & -3 & 20 & 0 \\ 0 & 0 & 27 & -7 & 2 \end{pmatrix}$$

CD $\notin$ LTD even though C $\in$ LTD and D $\in$ LTD

1.4. Result: The product of two tridiagonal matrices is pentadiagonal, i.e it has zeros when **|** i – j **|** > 2.

Proof : Let A = (a_{ij}) and B = (b_{ij}) where $a_{ij} = b_{ij} = 0$ if **|** i – j **|** >1

Then AB = (c_{ij}) where

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj_j}$ 1 $\leq$ i, j $\leq$ n

$= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots\ldots\ldots + a_{i,i-2}b_{i-2,j} + a_{i,1-1}b_{i-1,j} + a_{ii}b_{ij} + a_{i,i+1}b_{i+1,j} + \ldots\ldots + a_{in}b_{nj}$

case (i) j > i + 2

We have $a_{i1} = a_{i2} = \ldots\ldots\ldots\ldots\ldots\ldots a_{i,i-2} = 0$

$c_{ij} = a_{i,i-1}b_{i-1,j} + a_{ii}b_{ij} + a_{i,i+1}b_{i+1,j} + \ldots\ldots\ldots + a_{in}b_{nj}$

j – i-1 > 1 $\Rightarrow$ $b_{i+1,j}$ = 0

$j - i > 2 \Rightarrow b_{ij} = 0$

$j - i + 1 > 2 + 1 \Rightarrow b_{i-1,j} = 0$

Therefore $c_{ij} = 0$ if $j > i+2$

Similarly if $i > j + 2$, $c_{ij} = 0$

Therefore if $| j - i | > 2$, then $c_{ij} = 0$

1.5. Remark: The inverse of a nonsingular tridiagonal matrix is not necessarily tridiagonal.

Illustration :

$$A = \left(\begin{array}{ccc|cc} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 0 & 5 & 2 & 3 & 0 \\ \hline 0 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 6 & 5 \end{array}\right) = \left(\begin{array}{c|c} B & C \\ \hline E & D \end{array}\right)$$

We use the partition method [4] for computation of A^{-1}

Let $A^{-1} = \left(\begin{array}{c|c} X & Y \\ \hline Z & V \end{array}\right)$ where $AA^{-1} = I$

$V = (D - EB^{-1}C)^{-1}$

$Y = - B^{-1}CV$

$Z = -VEB^{-1}$

$X = B^{-1}(I - C Z)$

$$A^{-1} = \left(\begin{array}{ccc|cc} -\frac{9}{2} & \frac{33}{18} & \frac{1}{3} & \frac{-5}{14} & \frac{1}{14} \\ \frac{11}{4} & \frac{-33}{36} & \frac{-1}{6} & \frac{5}{28} & \frac{-1}{28} \\ \frac{5}{2} & \frac{-5}{6} & \frac{-1}{3} & \frac{5}{14} & \frac{-1}{14} \\ \hline \frac{-25}{4} & \frac{-25}{12} & \frac{5}{6} & \frac{-15}{28} & \frac{3}{28} \\ \frac{5}{2} & \frac{5}{2} & -1 & \frac{9}{14} & \frac{1}{14} \end{array}\right)$$

1.6. Method of finding the inverse of a nonsingular l t d matrix:

If L is a nonsingular l t d matrix of order n , L being a lower traingular matrix, its inverse L^{-1} is a lower traingular matrix.

Let L = (l_{ij}) and $L^{-1} = (x_{ij})$

where l_{ij} = 0 if | j - i| > 1 and

x_{ij} = 0 if j > i

$$\text{Then } LL^{-1} = \begin{pmatrix} l_{11} & 0 & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & 0 & \dots & 0 \\ 0 & l_{32} & l_{33} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \dots l_{n,n-1} & .l_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & 0 & 0 & \dots & 0 \\ x_{21} & x_{22} & 0 & \dots & 0 \\ x_{31} & x_{32} & x_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & \dots & x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots\ldots & & \ldots & \ldots & \ldots \\ \ldots\ldots & & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

$\Rightarrow l_{ii} \; x_{ii}$ = 1, $\forall$ i and

for i > j, 0 = (00………… $l_{i,i-1} \;\; l_{ii}$ 0 0…..0)$(0\ 0.....x_{jj} \;\; x_{j+1,j} \;\; x_{ij} ....x_{nj})^t$

$= l_{i,i-1}x_{i-1,j} + l_{ii}x_{ij}$

$$\Rightarrow x_{ij} = \frac{-l_{i,i-1}x_{i-1,j}}{l_{ii}}$$

New Recursive algorithm for finding the inverse of a nonsingular ltd matrix of order n :

1.6.1 Algorithm: Given L = (l_{ij}) where $l_{ii} \neq 0$, $\forall$ i and l_{ij} = 0 for j – i >1

Compute i = 1 to n, $x_{ii} = l_{ii}^{-1}$

For i = 2 to n and j = 1 to i – 1

Compute $x_{ij} = \dfrac{-l_{i,i-1} \;\; x_{i-1,j}}{l_{ii}}$

$$L^{-1} = \begin{pmatrix} x_{11} & 0 & 0 & \ldots & 0 \\ x_{21} & x_{22} & 0 & \ldots & 0 \\ x_{31} & x_{32} & x_{33} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n1} & x_{n2} & x & \ldots & x_{nn} \end{pmatrix}$$

Illustration:

Let L = $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$

Let $L^{-1} = (x_{ij})$ where $x_{ij} = 0$ for $j > i$, $1 \leq i, j \leq 5$ such that $LL^{-1} = I$

$x_{11} = 1$

$x_{21} = \dfrac{-l_{21} \quad x_{11}}{l_{22}} = -0.75$

$x_{22} = l_{22}^{-1} = 0.25$

$x_{31} = \dfrac{-l_{32} \quad x_{21}}{l_{33}} = 0.3$

$x_{32} = 0.1$, $x_{33} = 0.2$

$x_{41} = -0.9$, $x_{42} = 0.3$, $x_{43} = -0.6$, $x_{44} = 0.5$

$x_{51} = 0.128571428$, $x_{52} = -0.042857142$

$x_{53} = 0.085714285$, $x_{54} = -0.071428571$

$x_{55} = 0.142857142$

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.75 & 0.25 & 0 & 0 & 0 \\ 0.3 & -0.1 & 0.2 & 0 & 0 \\ -0.9 & 0.3 & -0.6 & 0.5 & 0 \\ 0.128571428 & -0.042857142 & 0.085714285 & -0.071428571 & 0.142857142 \end{pmatrix}$$

1.7. Counting Arithmetic Operations:

We now arrive at the number of arithmetic operations for finding inverse of a lower tridiagonal matrix L of order n by the recursive method of theorem (1.6)

Let S_i be the number of operations required for computation of inverse of a lower tridiagonal matrix of order i when the inverse of the submatrix of order i-1 is known. Then S = $\sum_{i=1}^{n} S_i$ is the total number of operations required for the inverse of a given matrix L of order n.

Let $L_1 = (l_{11})$ and $L_1^{-1} = (l_{11}^{-1})$

For i = 2(1)n

$\delta_{ij} = 0$ if $j < i$

$=1$ if $j = i$

$e_i^t = (\delta_{i1}, \delta_{i2}, \ldots\ldots\ldots\ldots \delta_{ii})$

$$L_i = \begin{pmatrix} L_{i-1} & oe_{i-1} \\ l_{i,i-1}e_{i-1}t & l_{ii} \end{pmatrix}$$

Then $L_i^{-1} = \begin{pmatrix} X_{i-1} & oe_{i-1} \\ x_{i,i-1}e_{i-1}t & x_{ii} \end{pmatrix}$

where $x_{ii} = l_{ii}^{-1}$

$$x_{ij} = \frac{-l_{i,i-1}\ x_{i-1,j}}{l_{ii}}$$

$$L_i^{-1} = X_i$$

In this computation we don't consider the change of sign as an arithmetical operation.

TABLE 1.7.1

Computation of	No. of arithmetic operations	Total number of arithmetic operations
X_{ii} for i =1(1)n	n	n^2
X_{ij}, for i = 2(1) n j = 1 to i - 1	$\sum_{i=2}^{n} 2(i-1) = n^2 - n$	

Total number of computations, S = n^2

CHAPTER – 2

L U Decomposition for tridiagonal matrices:

2.1 Theorem [3]: Let A = (a_{ij}) be a matrix whose leading submatrices are nonsingular.

Then A has LU decomposition.

Proof: The proof is through mathematical induction.

Clearly this holds for n = 1 for a_{11} = (l_{11}) (u_{11}) where if u_{11} is prescribed arbitrarily l_{11} may be determined by $l_{11} = a_{11}/u_{11}$. Assume the theorem to be true for (n-1).

Let A = $\begin{pmatrix} A_{n-1} & b \\ c & a_{nn} \end{pmatrix}$ be a square matrix of order n.

By induction hypothesis, A_{n-1} has LU-decomposition $A_{n-1} = L_{n-1} U_{n-1}$, where L_{n-1} is a lower traingular matrix while U_{n-1} is a upper triangular matrix. Moreover, the assumption that A_{n-1} is nonsingular implies L_{n-1}, U_{n-1} are nonsingular matrices.

Set x = cU_{n-1}^{-1} $y = L_{n-1}^{-1}$b

and let l_{nn}, u_{nn} be any numbers such that

b = $L_{n-1}\, y$, c= $x\, U_{n-1}$

and $xy + l_{nn}\, u_{nn} = a_{nn}$

$\therefore\ l_{nn}\, u_{nn} = a_{nn} - xy$

Write L = $\begin{pmatrix} L_{n-1} & 0 \\ x & l_{nn} \end{pmatrix}$ and U = $\begin{pmatrix} U_{n-1} & y \\ 0 & u_{nn} \end{pmatrix}$

Then LU = $\begin{pmatrix} L_{n-1} & 0 \\ x & l_{nn} \end{pmatrix} \begin{pmatrix} U_{n-1} & y \\ 0 & u_{nn} \end{pmatrix}$ = A

By induction the result follows.

Example 2.2.1.

$$\begin{pmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 4 & 0 \\ -3 & 6 & -2 \end{pmatrix} \begin{pmatrix} -1 & 1 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 2 & -4 & 0 \\ 3 & -6 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

2.2. Uniqueness of the LU – Decomposition:

That the LU – decomposition is in general not necessarily unique is clear from the following,

Example 2.2.1.

$$\begin{pmatrix} -1 & 1 & 0 \\ 2 & 4 & 2 \\ 0 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 0 & 3/2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & 2/3 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 1/2 & 4 \end{pmatrix} \begin{pmatrix} -1/2 & -1/2 & 0 \\ 0 & 6 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear from the proof of theorem (2.1), that one of the leading diagonal elements l_{ii} or u_{ii} may be prescribed arbitrarily for the unique determination of the other. Generally, we impose the condition that u_{ii}=1, ie the all diagonal elements of the upper triangular matrix are prescribed to be unity. In such a situation the LU- decomposition will be unique. We now present our results on LU decomposition for tridiagonal matrices.

2.3. Result : If A is tridiagonal matrix and has nonsingular leading principal minors, then the L, U in any LU decomposition for A are ltd and utd matrices respectively.

Proof : since each leading principal minor is non-singular, the L and U in traingular decomposition for each minor are nonsingular. Hence the entries in the diagonals of L and U, are non-zero. In the general case we have many choices for l_{ii} and u_{ii}, from equations of the type $l_{ii}\ u_{ii}$ = k_i . However when we impose the condition that u_{ii} = 1 $\forall$ i, and hence L_i and U_i are uniquely fixed. Hence we assume that u_{ii} =1 $\forall$i.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \dots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n,n-1} & \dots & a_{nn} \end{pmatrix}$$

$$\text{and } L = \begin{pmatrix} l_{11} & 0 & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & l_{n3} & \dots & \dots & l_{nn} \end{pmatrix} \quad U = \begin{pmatrix} 1 & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & 1 & u_{23} & \dots & u_{2n} \\ 0 & 0 & 1 & \dots & u_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

such that A = LU

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & 1 & u_{23} & \dots & u_{2n} \\ 0 & 0 & 1 & \dots & u_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & \dots & l_{11}u_{1n} \\ l_{21} & l_{22}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} & \dots & l_{21}u_{1n} + l_{22}u_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ l_{n1} & l_{n1}u_{12} + l_{n2} & l_{n1}u_{13} + l_{n2}u_{23} + l_{n3} & \dots & \sum_{i=1}^{n-1} l_{ni}u_{in} + l_{nn} \end{pmatrix}$$

$l_{11} = a_{11}$, $l_{11}\ u_{12} = a_{12} \Rightarrow u_{12} = a_{12} / l_{11}$, $l_{11}\ u_{13} = 0 \Rightarrow u_{13} = 0$

Similarly $u_{14} = u_{15} =$…………..$= u_{1n} = 0$

$l_{21} = a_{21}$, $l_{2l}u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21}u_{12}$

$u_{23} = a_{23} / l_{22}$, $l_{21}u_{14} + l_{22}u_{24} = 0 \Rightarrow u_{24} = 0$

Similarly $u_{25} = u_{26} = \dots\dots\dots\dots = u_{2n} = 0$

Hence $l_{11} = a_{11}, u_{ij} = a_{ij} / l_{ii}, l_{ij} = a_{ij} - \sum_{k=1}^{i-2} l_{ik}u_{kj}$ when | i – j | = 1

$l_{ii} = a_{ii} - \sum_{k=1}^{i-1} l_{ik} u_{ki}$ if $1 \le i \le n$, $2 \le j \le n$.

If i – j > 1, i = 3 to n, j = 1 to n, then (i, j)th element in the product LU is

$l_{i1}u_{1j} + l_{i2}u_{2j} + ... + l_{i,i-3}u_{i-3,j} + l_{ij}u_{jj} = 0$

Since $l_{i1} = l_{i2} = ... = l_{i,i-3}$ = 0, $l_{ij}(1) = 0 \Rightarrow l_{ij} = 0$

Similarly if j – i > 1, j = 3 to n, i =1(1)n, the (i,j)th element in the product LU is

$l_{i1}u_{1j} + l_{i2}u_{2j} + ... + l_{i,j-3}u_{j-3,j} + l_{ii}u_{ij} = 0$

Since $u_{1j} = u_{2j} = ... = u_{j-3,j} = 0$ $l_{ii}u_{ij} = 0 \Rightarrow u_{ij} = 0$

Hence L in the product L U is a lower tridiagonal matrix

i.e L = (l_{ij}) if i – j > 1

and U in the product LU is an upper tridiagonal matrix

ie U = (u_{ij}) if j – i > 1

2.4. LU Decomposition of a tridiagonal matrix by single bordering:

Let A_n be tridiagonal matrix of order n whose leading principal minors are nonsingular and e_i be the column vector with i components having 1 in ith position and zero elsewhere. Let A_{n-1} be leading principal minor of order n –1 having ltd, utd decomposition, $A_{n-1} = L_{n-1}\ U_{n-1}$ where the diagonal in U_{n-1} consists of 1's alone.

Proof: For $1 \le i \le n-1$, let e_i = $(0,0,……..1)^t$ with i components

let $A_n = L_n U_n$ where

$$A_n = \begin{pmatrix} A_{n-1} & a_{n-1,n}e_{n-1} \\ a_{n,n-1}e_{n-1}{}^t & a_{nn} \end{pmatrix}, \quad L_n = \begin{pmatrix} L_{n-1} & 0e_{n-1} \\ l_{n,n-1}e_{n-1}{}^t & l_{nn} \end{pmatrix}$$

$$U_n = \begin{pmatrix} U_{n-1} & u_{n-1,n}e_{n-1} \\ 0e_{n-1}^t & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_{n-1} & a_{n-1,n}e_{n-1} \\ a_{n,n-1}e_{n-1}{}^t & a_{nn} \end{pmatrix} = \begin{pmatrix} L_{n-1} & 0e_{n-1} \\ l_{n,n-1}e_{n-1}{}^t & l_{nn} \end{pmatrix} \begin{pmatrix} U_{n-1} & u_{n-1,n}e_{n-1} \\ 0e_{n-1}^t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} L_{n-1}U_{n-1} & u_{n-1,n} \quad l_{n-1,n-1} \quad e_{n-1} \\ l_{n,n-1}u_{n-1,n-1} \quad e_{n-1}{}^t & l_{n,n-1} \; u_{n-1,n} + l_{nn} \end{pmatrix}$$

$\Rightarrow A_{n-1} = L_{n-1} \; U_{n-1}$

$a_{n-1,n}e_{n-1} = u_{n-1,n}l_{n-1,n-1}e_{n-1}$

$a_{n,n-1}e^t{}_{n-1} = l_{n,n-1}u_{n-1,n-1}e_{n-1}^t$

$a_{nn} = l_{n,n-1}u_{n-1,n} + l_{nn}$

$\Rightarrow u_{n-1,n} = \dfrac{a_{n-1,n}}{l_{n-1,n-1}}$

since $u_{ii} = 1$ for i = 1(1) n

$l_{n,n-1} = a_{n,n-1}$

$l_{nn} = a_{nn} - l_{n,n-1}u_{n-1,n}$, therefore $A_n = L_nU_n$

Hence A = LU

2.5. Algorithm for finding LU decomposition for tridiagonal matrix A = (a_{ij}) of order n by single bordering:

Write $A_1 = (a_{11}), \; l_{11} = a_{11}$

$L_1 = (l_{11})$

For i = 2(1)n

j = 1(1) i

e_{i-1} = ($\delta_{i-1,1}$ $\delta_{i-1,2}$, ...$\delta_{i-1,\,i-1}$)^t where δ_{ij} = 0, if j < i

= 1, if j = i

Write $A_i = \begin{pmatrix} A_{i-1} & a_{i-1,i}\ e_{i-1} \\ a_{i,i-1}\ e^t{}_{i-1} & a_{ii} \end{pmatrix}$

$u_{i-1,i} = \dfrac{a_{i-1,i}}{l_{i-1,i-1}}$

$l_{i,i-1} = a_{i,i-1}$

$l_{ii} = a_{ii} - l_{i,i-1} u_{i-1,i}$

$L_i = \begin{pmatrix} L_{i-1} & 0\ e_{i-1} \\ l_{i,i-1} e^t{}_{i-1} & l_{ii} \end{pmatrix}$, $U_i = \begin{pmatrix} U_{i-1} & U_{i-1,i}\ e_{i-1} \\ 0\ e^t{}_{i-1} & u_{ii} \end{pmatrix}$

A = $A_n = L_n\ U_n$

$\therefore$ A = LU

2.5.1. Ilustration:

A = $\begin{pmatrix} 2 & 3 & 0 & 0 & 0 \\ 1 & -1 & -2 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 6 & 1 & -4 \\ 0 & 0 & 0 & 3 & -1 \end{pmatrix}$

Step 1: i = 1

Let A_1 = (2) and L_1 = (l_{11}),U_1= (1) such that

$L_1 U_1 = A_1$

$l_{11} = a_{11}$ = 2, u_{11} = 1

Step 2: i =2

Let $A_2 = \begin{pmatrix} A_1 & 3 \\ 1 & -1 \end{pmatrix}$

$u_{12} = \dfrac{a_{12}}{l_{11}}$ = 1.5

$l_{21} = a_{21}$ = 1

$l_{22} = a_{22} - l_{21}\, u_{12}$ = -2.5

$L_2 = \begin{pmatrix} 2 & 1 \\ 0 & -2.5 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 1 & 1.5 \\ 0 & 1 \end{pmatrix}$

$L_2 \; U_2 = A_2$

Step 3: i = 3

Let $A_3 = \begin{pmatrix} A_2 & -2e_2 \\ 1e_2^t & 3 \end{pmatrix}$

$l_{23} = \dfrac{a_{23}}{l_{22}}$ = 0.8

$l_{32} = a_{32}$ = 1

$l_{33} = a_{33} - l_{32}\, u_{23}$ = 2.2

$L_3 = \begin{pmatrix} L_2 & 0e_2 \\ 1e_2^t & 2.2 \end{pmatrix}$ and $U_3 = \begin{pmatrix} U_2 & 0.8e_2 \\ 0e_2^t & 1 \end{pmatrix}$

$L_3 U_3 = A_3$

Step 4: i = 4 $\quad u_{34} = \dfrac{a_{34}}{l_{33}}$ = -0.909090909

$l_{43} = a_{43}$ = 6

$l_{44} = a_{44} - l_{43}\; u_{34}$ = 6.454545455

$$L_4 = \begin{pmatrix} L_3 & 0e_3 \\ 6e_3^t & 6.454545455 \end{pmatrix} \text{ and } U_4 = \begin{pmatrix} U_3 & -0.909090909e_3 \\ 0e_3^t & 1 \end{pmatrix}$$

$$L_4 U_4 = A_4$$

Step 5: i = 5

$u_{45} = \dfrac{a_{45}}{l_{44}}$ = -0.619718309

$l_{54} = a_{54}$ = 3

$l_{55} = a_{55} - l_{54}\ u_{45}$ = 0.859154929

$$L_5 = \begin{pmatrix} L_4 & 0e_4 \\ 3e_4^t & 0.859154929 \end{pmatrix} \text{ and } U_5 = \begin{pmatrix} U_4 & -0.619718309e_4 \\ 0\,e_4^t & 1 \end{pmatrix}$$

$$L_5 U_5 = A_5$$

Finally A = $A_5 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & -2.5 & 0 & 0 & 0 \\ 0 & 1 & 2.2 & 0 & 0 \\ 0 & 0 & 6 & l_4 & 0 \\ 0 & 0 & 0 & 3 & l_5 \end{pmatrix} \begin{pmatrix} 1 & 1.5 & 0 & 0 & 0 \\ 0 & 1 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & u_3 & 0 \\ 0 & 0 & 0 & 1 & u_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

where l_4 = 6.454545455, l_5 = 0.859154929

u_3 = 0.909090909, u_4 = -0.619718309

Illustration 2.5.2.

A = $\begin{pmatrix} 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 6 & 0 & -4 \\ 0 & 0 & 0 & 3 & -1 \end{pmatrix}$

As above we get

L = $\begin{pmatrix} l_{11} & 0 & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 & 0 \\ 0 & 0 & l_{43} & l_{44} & 0 \\ 0 & 0 & 0 & l_{54} & l_{55} \end{pmatrix}$ and U = $\begin{pmatrix} 1 & u_{12} & 0 & 0 & 0 \\ 0 & 1 & u_{23} & 0 & 0 \\ 0 & 0 & 1 & u_{34} & 0 \\ 0 & 0 & 0 & 1 & u_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

where l_{11} = 2 u_{12} = 1.5

l_{21} =1, l_{22} = -1.5

l_{32} =1, l_{33} = 1.666666667, u_{23} = 1.333333333

l_{43} = 6, l_{44}= 7.2 u_{34} = -1.2

l_{54} = 3, l_{55} = 0.666666666 u_{45} = =0.555555555

LU = A

2.6. This algorithm is valid only when all the principal minors are nonsingular. Even otherwise one may get such a L and U so that A = LU, even though the algorithm fails.

Illustration:

A= $\begin{pmatrix} 4 & 6 & 0 & 0 & 0 \\ 6 & 9 & 12 & 0 & 0 \\ 0 & 0 & 19 & 5 & 0 \\ 0 & 0 & 8 & -5 & -6 \\ 0 & 0 & 0 & -3 & 18 \end{pmatrix}$

Step1: i =1

Let A_1 = (4) , l_{11} = a_{11} =4, u_{11} = 1

L_1 = $(l_{11}), U_1$ = (1)

Step 2: i =2

A_2 = $\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$

l_{21} = a_{21} = 6

u_{12} = $\frac{a_{12}}{l_{11}} = \frac{6}{4} = \frac{3}{2}$

l_{22} = a_{22} − l_{21} u_{12}

= 9-6 (3/2) = 0

u_{22} = 1

L_2 = $\begin{pmatrix} L_1 & 0 \\ l_{21} & l_{22} \end{pmatrix}$ U_2 = $\begin{pmatrix} U_1 & u_{12} \\ 0 & u_{22} \end{pmatrix}$

Step 3: i = 3

A_3 = $\begin{pmatrix} A_2 & 12e_2 \\ oe_2{}^t & 19 \end{pmatrix}$

l_{32} = a_{32} = 0

l_{22} u_{23} = a_{23} =12

This is not possible

However we have A = LU

Where L = $\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 3 & 5 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$, U = $\begin{pmatrix} 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$

2.7. Counting Arithmetic operations:

We now find the number of arithmetic operations for finding LU decomposition for a tridiagonal matrix A of order n by the recursive method of theorem (2.4).

Let S_i be the number of operations required for computation of LU decomposition for a matrix of order i when LU decomposition of the submatrix of order i –1 is known. Then S = $\sum_{i=1}^{n} S_i$ is the total number of operations required for the LU decomposition of a given matrix A of order n.

Let $A_1 = (a_{11}) = L_1 U_1$ where $L_1 = (a_{11}), U_1 = (1)$

for i = 2(1) n

$A_i = L_i U_i$ where $L_i = \begin{pmatrix} L_{i-1} & 0e_{i-1} \\ l_{i,i-1}e_{i-1}{}^t & l_{ii} \end{pmatrix}$ and $U_i = \begin{pmatrix} U_{i-1} & U_{i,i-1}e_{i-1} \\ 0e^t{}_{i-1} & 1 \end{pmatrix}$

$u_{i-1,i} = \dfrac{a_{i-1,i}}{l_{i-1,i-1}}$

$$l_{i,i-1} = a_{i,i-1}$$

$l_{ii} = a_{ii} - l_{i,i-1}u_{i-1,i}$

In this computation we don't consider the change of sign as an arithmetical operation.

TABLE 2.7.1

Computation of	No. of Arithmetic Operations	Total number of arithmetic operations
L_i	2	3
U_i	1	

Total number of operations = 3+ 3 + ……………(n-1) times = 3(n-1)

Note : If we assume that u_{ii} = 1 $\forall$i, then the representation is unique and the number of arithmetic operations is 3(n-1).

Comparison:

The arithmetic operations required for computation of L and U in A = LU are counted in [7] and are found to be (**1/6)n(n-1)(4n+1),** where n is the order of A .

In the case of a tridiagonal matrix where our algorithm (2.5) is applied, this number reduces to **3(n-1).**

2.8. LU Decomposition of a symmetric, positive definite and tridiagonal matrix A by single bordering:

Let A = A_n be symmetric, positive definite and tridiagonal matrix of order n and e_i be column vector of order i having 1 in ith position and 0 elsewhere. Let A_{n-1} be the leading principal minor of A_n of order n-1. Assume that $A_{n-1} = L_{n-1}L^t{}_{n-1}$ where L_{n-1} is ltd (2.3) then A = LL^t

where L = $\begin{pmatrix} L_{n-1} & 0\,e_{n-1} \\ l_{n,n-1}e_{n-1}{}^{t} & l_{nn} \end{pmatrix}$

$l_{n,n-1} = \dfrac{a_{n,n-1}}{l_{n-1,n-1}}$

$l_{nn} = (a_{nn} - l^2{}_{n,n-1})^{1/2}$

Proof: LL^t = $\begin{pmatrix} L_{n-1} & 0\,e_{n-1} \\ l_{n,n-1}e_{n-1}{}^{t} & l_{nn} \end{pmatrix}\begin{pmatrix} L^t{}_{n-1} & l_{n,n-1}e_{n-1} \\ 0e^t{}_{n-1} & l_{nn} \end{pmatrix}$

= $\begin{pmatrix} L_{n-1}L^t_{n-1} & l_{n,n-1}L_{n-1} \quad e_{n-1} \\ l_{n,n-1}e_{n-1}{}^{t}L^t_{n-1} & l^2{}_{n,n-1}\ e^t{}_{n-1}\ e_{n-1} \end{pmatrix}$

= $\begin{pmatrix} A_{n-1} & a_{n-1,n}\ e_{n-1} \\ a_{n,n-1}e_{n-1}{}^{t} & a_{nn} \end{pmatrix}$ = A

$A_{n-1} = L_{n-1}L_{n-1}{}^{t}$

$a_{n-1,n}e_{n-1} = l_{n,n-1}l_{n-1,n-1}e_{n-1}$

$a_{n,n-1}e^t_{n-1} = l_{n,n-1}l_{n-1.n-1}e^t_{n-1}$

$a_{nn} = l^2_{n,n-1} + l^2_{nn}$

Since A is symmetric a_{ij} = a_{ji} for 1≤ i, j ≤ n

$a_{n-1,n} = a_{n,n-1}$

$l_{n,n-1}$= $\dfrac{a_{n,n-1}}{l_{n-1,n-1}}$

l_{nn} = ($a_{nn} - l^2_{n,n-1})^{1/2}$

Hence LL^T = $\begin{pmatrix} A_{n-1} & a_{n-1,n} \quad e_{n-1} \\ a_{n,n-1}\ e^t_{n-1} & a_{nn} \end{pmatrix}$ = A

2.9. Algorithm for finding the LU Decomposition of a symmetric, positive definite and tridiagonal matrix A = (a_{ij}) of order n:

Assume A = (a_{ij}) is symmetric, positive definite and tridiagonal marix of order n.

Write $A_1 = (a_{11})$, $l_{11} = a_{11}^{1/2}$

$L_1 = (l_{11})$

for i =2(1)n, j = 1(1)i

$e_i^t = (\delta_{i1}, \delta_{i2}, \ldots\ldots\ldots\delta_{ii})$, $\delta_{ij} = 0$ if j < i

= 1 if j = i

write $A_i = \begin{pmatrix} A_{i-1} & a_{i-1,i} \; e_{i-1} \\ a_{i,i-1} \, e_{i-1}^t & a_{ii} \end{pmatrix}$

$$l_{i,i-1} = \frac{a_{i,i-1}}{l_{i-1,i-1}}$$

$$l_{ii} = (a_{ii} - l_{i,i-1}^2)^{1/2}$$

$$L_i = \begin{pmatrix} L_{i-1} & 0 \, e_{i-1} \\ l_{i,i-1} \, e_{i-1}^t & l_{ii} \end{pmatrix}$$

A = $A_i = L_i L_i^t$

$\therefore$ A = LL^t

2.9.1. Illustration:

$$A = \begin{pmatrix} 15 & 4 & 0 & 0 & 0 \\ 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 9 & 6 & 0 \\ 0 & 0 & 6 & 10 & 3 \\ 0 & 0 & 0 & 3 & 4 \end{pmatrix}$$

A is symmetric, positive definite and tridiagonal matrix of order 5.

Step 1: i = 1

Let A_1 = (15), l_{11} = $(15)^{1/2}$ = 3.872983346

$\therefore$ L_1 = (3.872983346)

Step2 : i = 2

$$A_2 = \begin{pmatrix} A_1 & 4 \\ 4 & 7 \end{pmatrix}$$

$$l_{21} = \frac{a_{21}}{l_{11}} = 1.032795559$$

$$l_{22} = (a_{22} - l_{21}^2)^{1/2} = 2.435843454$$

$$\therefore L_2 = \begin{pmatrix} L_1 & 4 \\ 1.032795559 & 2.435843454 \end{pmatrix}$$

Step 3: i = 3

$$A_3 = \begin{pmatrix} A_2 & 1e_2 \\ 1e_2^t & 9 \end{pmatrix}$$

$$L_{32} = \frac{a_{32}}{l_{22}} = 0.410535413$$

$$l_{33} = (a_{33} - l_{32}^2)^{1/2} = 2.971777359$$

$$L_3 = \begin{pmatrix} L_2 & 0e_2 \\ 0.410535413e_2^t & 2.971777359 \end{pmatrix}$$

Step 4: i =4

$$A_4 = \begin{pmatrix} A_3 & 6e_3 \\ 6e_3^t & 10 \end{pmatrix}$$

$l_{43} = \dfrac{a_{43}}{l_{33}}$ = 2.018993779

$l_{44} = (a_{44} - l_{43}^2)^{1/2}$ = 2.433857868

$$L_4 = \begin{pmatrix} L_3 & 0e_3 \\ 2.018993779e_3^t & 2.433857868 \end{pmatrix}$$

Step 5: i = 5

$$A_5 = \begin{pmatrix} A_4 & 3e_4 \\ 3e_4^t & 4 \end{pmatrix}$$

$l_{54} = \dfrac{a_{54}}{l_{44}}$ = 1.232611008

$l_{55} = (a_{55} - l_{54}^2)^{1/2}$ = 1.575014218

$$L_5 = \begin{pmatrix} L_4 & Oe_4 \\ 1.232611008e_4^t & 1.575014318 \end{pmatrix}$$

A= $A_5 = L_5 L_5^t$

where $L_5 = \begin{pmatrix} l_{11} & 0 & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 & 0 \\ 0 & 0 & l_{43} & l_{44} & 0 \\ 0 & 0 & 0 & l_{54} & l_{55} \end{pmatrix}$

l_{11} = 3.872983346

l_{21} = 1.032795559

l_{22} = 2.435843454

l_{32} = 0.410535413

l_{33} = 2.971777359

l_{43} = 2.018993779

l_{44} = 2.433857868

l_{54} = 1.232611008

l_{55} = 1.575014318

2.10. Counting Arithmetic Operations:

We now find the number of arithmetic operations for finding LL^t decomposition of a symmetric, positive definite and tridiagonal matrix A of order n by the recursive method of theorem (2.8).

Let S_i be the number of operations required for computation of LL^t decomposition of a matrix of order i when LL^t decomposition of the submatrix of order i-1 is known. Then S = $\sum_{i=1}^{n} S_i$ is the total number of operations required for the LL^t decomposition of a given matrix A of order n.

Let $A_1 = (a_{11}) = L_1 L_1^t$ where $L_1 = (a_{11}{}^{1/2})$

for i =2(1)n

$A_i = L_i L_i^t$ where $L_i = \begin{pmatrix} L_{i-1} & 0e_{i-1} \\ l_{i,i-1}e_{i-1}^t & l_{ii} \end{pmatrix}$

and $l_{i,i-1} = \dfrac{a_{i,i-1}}{l_{i-1,i-1}}$

$l_{ii} = (a_{ii} - l_{i,i-1}^2)^{1/2}$

In this computation we don't consider the change of sign as an arithmetical operation.

TABLE 2.10.1

Computation of	No. of arithmetic operations	Total number of arithmetic operations
$l_{i,i-1}$	1	4
l_{ii}	3	
		$S_i = 4$

S_1 = total number of arithmetic operations of L_1 = 1

S = $S_1 + \sum_{i=2}^{n} S_i$

= 1+4+4+………………(n-1) times

= 1+4(n-1) = 4n-3

CHAPTER – 3

In this section we consider the special type of ltd matrices in which $l_{2k,2k-1}=0\ \forall k$.

$$\text{Illustration : } A = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 5 \end{pmatrix}$$

3.1. Result: The set SLTD of all (nxn) ltd matrices in which (2k,2k-1) entries are zero $\forall$ k is a vector space of dimension p, where

p = $\frac{3n-1}{2}$, if n is odd and $\frac{3n-2}{2}$, if n is even.

This vector space is a ring with identity with respect to matrix multiplication.

Proof:

Step 1: clearly O $\in$ SLTD. So SLTD $\neq \phi$

Let A,B $\in$ SLTD, $\alpha \in$ R

where A = (a_{ij}), B = (b_{ij}) in which $a_{2k,2k-1} = b_{2k,2k-1} = 0\ \forall$ k and j – i > 1

$\Rightarrow$ A+B = $(a_{ij})+(b_{ij})= (a_{ij}+b_{ij})$

Since $a_{ij}=0$ and $b_{ij}=0$ if j – i > 1 and $a_{2k,2k-1}=b_{2k,2k-1}=0\ \forall$ k

$a_{ij}+b_{ij}=0$ if j – i > 1 and $a_{2k,2k-1}=b_{2k,2k-1}=0\ \forall$ k

Hence A+B $\in$ SLTD

Let $\alpha \in$ R
α A = $\alpha(a_{ij})$ = (αa_{ij})
Since $a_{ij}=0$ if j – i > 1 and $a_{2k,2k-1}=0\ \forall i$
$\alpha a_{ij}=0$ if j – i > 1 and $a_{2k,2k-1}=0\ \forall i$

Hence αA $\in$ SLTD

Therefore SLTD is a nonempty set and is closed under addition and scalar multiplication.

Hence SLTD is a vector space.

Write E_{ij} for the nxn matrix with 1 in the (i,j)th place and zero elsewhere.

Let B2 = { E_{ij} / $j - i > 1, 1 \leq i,\ j \leq n$ and (2k,2k-1) entries are zero $\forall$k}

then B2 $\subseteq$ TD and B2 is linearly independent.

Let A $\in$ SLTD, A=(a_{ij}) where $a_{ij} = 0$, if j – i > 1 and $a_{2k,2k-1} = 0 \ \forall k$

then A = $\sum_{i,j=1}^{n} a_{ij} E_{ij}$ when j – i > 1 and $a_{2k,2k-1} = 0 \ \forall k$

$\therefore$B2 spans SLTD.

Since B2 is a linearly independent subset of SLTD and spans SLTD, B2 is a basis for SLTD and dimension of SLTD = p

Step 2: SLTD is closed under matrix multiplication which is not necessarily commutative.

AB = (p_{ij}) where $p_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}$

$p_{2k+1,.2k} = a_{2k+1,2k} \quad b_{2k,2k} + a_{2k+1,2k+1} \quad b_{2k+1,2k}$

$p_{2k,2k-1} = 0$

$p_{ii} = a_{ii} b_{ii}$ where i = 1(1)n

$\therefore$AB$\in$ SLTD

Remark: If k>1, SLTD is not necessarily commutative with respect to multiplication.

Example : A = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ B = $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

(3,2) element in AB = 10

(3,2) element in BA = 7

3.2. Result: The inverse of a nonsingular matrix of SLTD is also a matrix of SLTD.

Let L $\in$ SLTD when L = (l_{ij}) in which $l_{2k,2k-1} = 0 \ \forall k$

Clearly the inverse L is lower triangular.

Let L^{-1} = $\begin{pmatrix} x_{11} & 0 & 0 & \dots & 0 \\ x_{21} & x_{22} & 0 & \dots & 0 \\ x_{31} & x_{32} & x_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{pmatrix}$

(2k,2k-1) element in the product LL^{-1} = I is

$$l_{2k,1} \ x_{1,2k-1} + l_{2k,2}, x_{2,2k-1} + ... + l_{2k,2k-1} \ \ x_{2k-1,2k-1} = 0$$

Since $x_{1,2k-1} = ... = x_{2k-2,2k-1} = 0$ and $x_{2k-1,2k-1} \neq 0$, it follows that $l_{2k,2k-1} = 0$

If L=L_n is a nonsingular ltd matrix of order n, the leading principal minors are also of the same type. This suggests a recursion formula for L^{-1}=L_n^{-1} in terms of the inverse of $L_{n-1}^{-1}$where $L_{n-1}$is the n-1 x n-1 matrix obtained by removing the nth row and nth column of L_n.

3.2. Method of finding the inverse of a ltd matrix of order n in which (2k,2k-1) entries are zero by single bordering:

Theorem: Let L be a ltd matrix of order n in which (2k,2k-1) entries are zero. If L is nonsingular then L^{-1} is a ltd matrix of order n in which (2k,2k-1) entries are zero.

Proof: Let $L_{n-1}^{-1} = X_{n-1}, e_{n-1}^{t} = (0,0,.......0,1)$ with n -1 components.

Let $L_n = \begin{pmatrix} L_{n-1} & 0e_{n-1} \\ l_{n,n-1}e_{n-1}^{t} & l_{nn} \end{pmatrix}$ and $L_n^{-1} = \begin{pmatrix} X_{n-1} & 0e_{n-1} \\ x_{n,n-1}e_{n-1}^{t} & x_{nn} \end{pmatrix}$

$L_n X_n = \text{I}$

$$\begin{pmatrix} L_{n-1} & 0e_{n-1} \\ l_{n,n-1}e_{n-1}^{t} & l_{nn} \end{pmatrix} \begin{pmatrix} X_{n-1} & 0e_{n-1} \\ x_{n,n-1}e_{n-1}^{t} & x_{nn} \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0e_{n-1} \\ 0e^{t}{}_{n-1} & 1 \end{pmatrix}$$

$L_{n-1} X_{n-1} == \text{I}$

$l_{nn} x_{nn} = 1$

$l_{n,n-1} e_{n-1}^{t} X_{n-1} + l_{nn} x_{n,n-1} e_{n-1}{}^{t} = 0$

$\Rightarrow x_{nn} = l_{nn}^{-1}$

$l_{n,n-1} x_{n-1,n-1} + l_{nn} x_{n,n-1} = 0$

$$x_{n,n-1} = \frac{-l_{n,n-1}}{l_{n-1,n-1} l_{nn}}$$

It follows that if the (2k,2k-1) term of L is zero, then the same is true for L_{n-1}^{-1}.

Moreover the (k,k-1) term of L_{n-1}^{-1} is $\dfrac{-l_{k,k-1}}{l_{k-1,k-1} l_{kk}}$

3.3. Algorithm for finding the inverse of a ltd matrix L of order n in which $l_{2k,2k-1} = 0 \;\; \forall k$:

Given L = (l_{ij}) is ltd matrix of order n where $l_{2k,2k-1} = 0 \;\; \forall k$

Write $L_1 = (l_{11})$, $\quad x_{11} = l_{11}^{-1}$

$L_1^{-1} = (x_{11})$

For i = 2(1)n, j = 1(1)i

e_i^t = $(\delta_{i1,}\delta_{i2,}................\delta_{ii})$ where $\delta_{ij} = 0,$ if j < i

=1, if j=i

Write Li = $\begin{pmatrix} L_{i-1} & oe_{i-1} \\ l_{i,i-1}e_{i-1}^t & l_{ii} \end{pmatrix}$

$x_{ii} = l_{ii}^{-1}$

$$x_{i,i-1} = \frac{-l_{i,i-1}x_{i-1,i-1}}{l_{ii}}$$

$$L_i^{-1} = \begin{pmatrix} L_{i-1}^{-1} & oe_{i-1} \\ x_{i,i-1}e_{i-1}{}^t & x_{ii} \end{pmatrix}$$

$L_i L_i^{-1} = \mathrm{I}$

$\therefore$ LL^{-1}=I

3.3.1. Illustration:

1. L = $\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 6 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 & 3 \end{pmatrix}$

Step (i) i =1

L_1=(2) = L_1^{-1} = (0.5)

Step (ii) i=2

Write L_2 = $\begin{pmatrix} L_1 & 0 \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$x_{22} = l_{22}^{-1} = 1$

$x_{21} = \frac{-l_{21}}{l_{11}l_{22}} = 0$

Therefore $L_2^{-1} = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}$

Step (iii) i =3

L_3 = $\begin{pmatrix} L_2 & 0e_2 \\ -4e_2^t & 6 \end{pmatrix}$ where $e_2^t = (0,1)$

$x_{33} = l_{33}^{-1} = 0.166666666$

$x_{32} = \frac{-l_{32}}{l_{22}l_{33}} = 0.666666666$

Therefore $L_3^{-1} = \begin{pmatrix} L_2^{-1} & oe_2 \\ 0.666666666e_2^t & 0.166666666 \end{pmatrix}$

Step (iv) i =4

L_4 = $\begin{pmatrix} L_3 & 0e_3 \\ 0e_3^t & 5 \end{pmatrix}$ where $e_3 = (0,0,1)^t$

Write $x_{44} = l_{44}^{-1} = 0.2$

$x_{43} = \frac{-l_{43}}{l_{33}l_{44}} = 0$

Therefore $L_4^{-1} = \begin{pmatrix} L_3^{-1} & oe_3 \\ 0e_3^t & 0.2 \end{pmatrix}$

Step (v) i =5

L_5 = $\begin{pmatrix} L_4 & 0e_4 \\ 7e_4^t & 3 \end{pmatrix}$ where $e_4 = (0 0\ 0\ 1)^t$

Write $x_{55} = l_{55}^{-1} = 0.333333333$

$$x_{54} = \frac{-l_{54}}{l_{55} l_{44}} = -0.466666666$$

Therefore $L_5^{-1} = \begin{pmatrix} L_4^{-1} & oe_4 \\ 0.466666666e_4^t & 0.333333333 \end{pmatrix}$

Hence $L^{-1} = L_5^{-1} = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0.666666666 & 0.166666666 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.466666666 & 0.333333333 \end{pmatrix}$

3.4. Counting Arithmetic Operations:

We now find the number of arithmetic operations for finding inverse of a lower tridiagonal matrix L of order n in which $l_{2k,2k-1} = 0$ by the recursive method of theorem (3.2).

Let S_i be the number of operations required for computation of inverse of a lower tridiagonal matrix of order i when inverse of the submatrix of order i –1 is known. Then S= $\sum_{i=1}^{n} S_i$ is the total number of operations required for the inverse of a given matrix L of order n.

Let $L_1 = (l_{11})$ $and\, L_1^{-1} = (l_{11}^{-1})$

for i =2(1)n

$\delta_{ij} = 0$ if j < i

=1 if j = i

$e_i^t = (\delta_{i1}, \delta_{i2}, \ldots\ldots\ldots\ldots\ldots\ldots \delta_{ii})$

$L_i = \begin{pmatrix} L_{i-1} & oe_{i-1} \\ l_{i,i-1} e_{i-1}^t & l_{ii} \end{pmatrix}$

then $L_i^{-1} = \begin{pmatrix} X_{i-1} & oe_{i-1} \\ x_{i,i-1} e_{i-1}^t & x_{ii} \end{pmatrix}$

where $x_{ii} = l_{ii}^{-1}$

$$x_{i,i-1} = \frac{-l_{i,i-1}}{l_{i-1,i-1}l_{ii}}$$

$L_i^{-1} = X_i$

In this computation we don't consider the change of sign as an arithmetical operation.

Since $l_{2k,2k-1} = 0$ and $x_{2k,2k-1} = 0$ total number of computations must be counted separately for both odd and even cases.

TABLE 3.4.1

Matrix	Computation of	No. of arithmetic operations	Total number of arithmetic operations
L_i^{-1}, i is odd	$x_{i,i-1}$ x_{ii}	2 1	3
L_i^{-1}	x_{ii}	1	1

Case (i) n is odd

Total number of operations, S= $S_1 + S_2 + \ldots\ldots\ldots\ldots + S_n$

= 1+($S_2 + S_4 + \ldots\ldots S_{n-1}$)+($S_3 + S_5 + \ldots\ldots\ldots S_n$)

= 1+1 $\left(\frac{n-1}{2}\right) + 3\left(\frac{n-1}{2}\right)$=2n-1

Case (ii) n is even

Total number of operations

S= S_1 + ($S_2 + S_4 + \ldots\ldots S_n$)+($S_3 + S_5 + \ldots\ldots\ldots S_{n-1}$)

= 1+1 $\left(\frac{n}{2}\right) + 3\left(\frac{n-2}{2}\right)$=2n-2

Comparison with a lower triangular matrix of order n in section 1.

The arithmetic operations required for computation of inverse of lower tridiagonal matrix of order n are counted and found to be n^2

In case of a lower tridiagonal matrix in which (2k,2k-1) entries are zero where our algorithm (3.3) is applied, this number reduces to

$$\begin{cases} 2n-1, \text{ when n is odd,} \\ 2n-2, \text{ when n is even.} \end{cases}$$

3.5. Result : If A is symmetric, positive definite and tridiagonal matrix of the type A = (a_{ij}) where $a_{2k,2k-1} = 0$ then A = LL^t where L is ltd and $l_{2k,2k-1} = 0$. For such a L, L^{-1} is also of the same type.

Proof : Let A = $$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 & \ldots & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & a_{43} & a_{44} & \ldots & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & a_{nn} \end{pmatrix}$$

and L = $$\begin{pmatrix} l_{11} & 0 & 0 & \ldots & 0 \\ l_{21} & l_{22} & 0 & \ldots & 0 \\ 0 & l_{32} & l_{33} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & l_{n,n-1} & l_{nn} \end{pmatrix}$$

such that A = LL^t

$$l_{11}^2 = a_{11} \Rightarrow l_{11} = a_{11}^{1/2}$$

$$l_{11} l_{21} = 0 \Rightarrow l_{21} = 0$$

$$l_{21}{}^2 + l_{22}{}^2 = a_{22} \Rightarrow l_{22} = a_{22}^{1/2}$$

$$l_{32} l_{22} = a_{32} \Rightarrow l_{32} = \frac{a_{32}}{l_{22}}$$

Therefore (2 i-1, 2 i-1) element

$$l^2{}_{2i-1,2i-2} + l^2{}_{2i-1,2i-1} = a_{2i-1,2i-1}$$

$$l^2{}_{2i-1,2i-1} = a_{2i-1,2i-1} - l^2{}_{2i-1,2i-2}$$

$$l_{2i-1,2i-1} = (a_{2i-1,2i-1} - l^2{}_{2i-1,2i-2})^{1/2}$$

$(2i-1\,,\,2i-2)$ element.

$$l_{2i-1,2i-2} \; l_{2i-2,2i-2} = a_{2i-1,2i-2}$$

$$l_{2i-1,2i-2} = \frac{a_{2i-1,2i-2}}{l_{2i-2,2i-2}}$$

$(2i, 2i-1)$ element.

$$l_{2i,2i-1} \; l_{2i-1,2i-1} + l_{2i,2i}(0) = 0$$

$$\therefore l_{2i,2i-1} = 0$$

$(2i, 2i)$ element.

$$l^2{}_{2i,2i} = a_{2i,2i} \Rightarrow l_{2i,2i} = (a_{2i,2i})^{1/2}$$

Hence $l_{2i,2i} = a_{2i,2i}^{1/2}$

$$l_{2i,2i-1} = 0$$

$$l_{2i+1,2i} = \frac{a_{2i+1,2i}}{l_{2i,2i}}$$

$$l_{2i+1,2i+1} = (a_{2i+1,2i+1} - l_{2i+1,2i}^2)^{1/2}$$

$$l_{11} = a_{11}^{1/2}$$

3.6 Single bordering for LL^t decomposition of a symmetric, positive definite and tridiagonal matrix A=(a_{ij}) of order n, where $a_{2k,2k-1} = 0 \ \forall k$:

Proof: Let $A_1 \ A_2 \ldots\ldots\ldots\ldots\ldots A_n$ be the leading principal minors of A.

$A_1 = (a_{11}) \quad and \quad L_1 = (l_{11})$ such that

$A_1 = L_1 L_1^T$

$l_{11} = a_{11}^{1/2}$

$\therefore L_1 = \left(a_{11}^{1/2}\right)$

for i = 2(l)n

Assume that $A_{i-1} = L_{i-1} L_{i-1}^t$ where (2k,2k-1) entries in the ltd matrix L_{i-1} are zero.

$e_i^t = (\delta_{i1}, \delta_{i2}, \ldots\ldots\ldots\ldots\ldots \delta_{ii})^t, \delta_{ij} = 0$ if j < i

= 1 if j = i

write $L_i = \begin{pmatrix} L_{i-1} & oe_{i-1} \\ l_{i,i-1} e_{i-1}^t & l_{ii} \end{pmatrix}$ ________(1)

Then $L_i L_i^t = A_i$ if

$A_{i-1} = L_{i-1} L_{i-1}^t$

since $a_{i-1,i} = a_{i,i-1}$

$l_{i,i-1} \ \ l_{i-1,i-1} = a_{i,i-1}$

$l^2{}_{i,i-1} + l_{ii}^2 = a_{ii}$

$\Rightarrow l_{i,i-1} = \dfrac{a_{i,i-1}}{l_{i-1,i-1}}$

$l_{ii}^2 = a_{ii} - l_{i,i-1}^2$

that is $l_{i,i-1} = \dfrac{a_{i,i-1}}{l_{i-1,i-1}}$

$l_{ii} = (a_{ii} - l_{i,i-1}^2)^{1/2}$

These values $l_{i,i-1}, l_{ii}$ give L_i by (1)

3.7. Algorithm:

Algorithm for finding LL^t decomposition of a symmetric, positive definite and tridiagonal matrix of order n in which (2k,2k-1) entries are zero ∀ k,

Assume A=(a_{ij}) is symmetric, positive definite and tridiagonal matrix of order n in which $a_{2k,2k-1} = 0 \quad \forall k$

Write $A_1 = (a_{11}), l_{11} = a_{11}^{1/2}$

L_1= (l_{11})

for i = 2(1)n, j = 1(1) i

$e_i^t = \left(\delta_{i1}, \delta_{i2} ....\delta_{ii}\right)$, $\delta_{ij} = 0,$ if j < i

= 1, if j = i

write $A_i = \begin{pmatrix} A_{i-1} & a_{i-1,i}\ e_{i-1} \\ a_{i,i-1} e_{i-1}^t & a_{ii} \end{pmatrix}$

$l_{i,i-1} = \dfrac{a_{i,i-1}}{l_{i-1,i-1}}$

$l_{ii} = (a_{ii} - l_{i,i-1}^2)^{1/2}$

$L_i = \begin{pmatrix} L_{i-1} & oe_{i-1} \\ l_{i,i-1} e_{i-1}^t & l_{ii} \end{pmatrix}$

A= $A_i = L_i L_i^t$

A = LL^t

Illustration:

$$A = \begin{pmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & 7 & 4 & 0 & 0 \\ 0 & 4 & 9 & 0 & 0 \\ 0 & 0 & 0 & 10 & 3 \\ 0 & 0 & 0 & 3 & 4 \end{pmatrix}$$

Step 1: i = 1

let $A_1 = (15)$, $l_{11} = 15^{1/2}$ = 3.872983346

$L_1 = (l_{11}) = (3.872983346)$

Step 2: i = 2

$$A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & 7 \end{pmatrix}$$

$$l_{21} = \frac{a_{21}}{l_{11}} = 0$$

$$l_{22} = (a_{22} - l_{21}^2)^{1/2} = 2.645751311$$

Therefore $L_2 = \begin{pmatrix} L_1 & 0 \\ 0 & 2.645751311 \end{pmatrix}$

Step 3: i = 3

$A_3 = \begin{pmatrix} A_2 & 4e_2 \\ 4e_2^t & 9 \end{pmatrix}$ where $e_2 = (0,1)^t$

$$l_{32} = \frac{a_{32}}{l_{22}} = 1.511857892$$

$$l_{33} = (a_{33} - l_{32}^2)^{1/2} = 2.591193878$$

Therefore $L_3 = \begin{pmatrix} L_2 & 0e_2 \\ 1.511857892e_2^t & 2.591193878 \end{pmatrix}$

Step 4: i = 4

A_4 = $\begin{pmatrix} A_3 & 0e_3 \\ 0e_3^t & 10 \end{pmatrix}$ where $e_3 = (0,0,1)^t$

$$l_{43} = \frac{a_{43}}{l_{33}} = 0$$

$$l_{44} = (a_{44} - l_{43}^2)^{1/2} = 3.16227766$$

$$\therefore L_4 = \begin{pmatrix} L_3 & 0e_3 \\ 0e_3^t & 3.16227766 \end{pmatrix}$$

Step 5: i = 5

A_5 = $\begin{pmatrix} A_4 & 3e_4 \\ 3e_4^t & 4 \end{pmatrix}$ where $e_4 = (0,0,0,1)^t$

$$l_{54} = \frac{a_{54}}{l_{44}} = 0.948683298$$

$$l_{55} = (a_{55} - l_{54}^2)^{1/2} = 1.760681686$$

$$\therefore L_5 = \begin{pmatrix} L_4 & 0e_4 \\ 0.948683298e_4^t & 1.760681686 \end{pmatrix}$$

$\therefore$ A = $A_5 = L_5 L_5^t$

A = LL^t

Where

$$L = \begin{pmatrix} 3.872983346 & 0 & 0 & 0 & 0 \\ 0 & 2.645751311 & 0 & 0 & 0 \\ 0 & 1.511857892 & 2.591193878 & 0 & 0 \\ 0 & 0 & 0 & 3.16227766 & 0 \\ 0 & 0 & 0 & 0.948683298 & 1.760681686 \end{pmatrix}$$

3.8. Counting Arithmetic Operations:

We now find the number of arithmetic operations for finding LL^t decomposition of a symmetric, positive definite and tridiagonal matrix A of order n in which $a_{2k,2k-1}$= 0, $\forall$ k by the recursive method of theorem [3.5] .

Let S_i be the number of operations required for computation of LL^t decomposition of a tridiagonal matrix of order i when LL^t decomposition of the submatrix of order i=1 is known. Then S= $\sum_{i-1}^{n} S_i$ is the total number of operations required for the LL^t decomposition of a given matrix A of order n. Since $a_{2k,2k-1}$=0, S can be computed separately for both even and odd cases.

Let $A_1 = (a_{11})$ and $A_1 = L_1 L_i^t$ where $L_1 = (l_{11})$
for i = 2(1)n

$$A_i = L_i L_i^T \text{ where } L_i = \begin{pmatrix} L_{i-1} & oe_{i-1} \\ l_{i,i-1} e_{i-1}^t & l_{ii} \end{pmatrix}$$

And $l_{2k,2k-1} = 0$

$$l_{i,i-1} = \frac{a_{i,i-1}}{l_{i-1,i-1}}$$

$$l_{ii} = (a_{ii} - l_{i,i-1}^2)^{1/2} \text{ where } a_{ii} - l_{i,i-1}^2 > 0$$

$\therefore A_i = L_i L_i^T$ for each i

In this computation we don't consider the change of sign as an arithmetical operation.

TABLE 3.8.1

Matrix	Computation of	No. of arithmetic operations	Total no. of arithmetic operations
$L_{i,}\, i$ is odd	$l_{i,i-1}$ l_{ii}	1 3	4
L_i, i is even	l_{ii}	3	3

Case (i) n is odd:

Total number of operations:

S= S_1 + ($S_2 + S_4 + ...S_{n-1}$)+($S_3 + S_5 + ...S_n$)

$$= 1+ 3\left(\frac{n-1}{2}\right) + 4\left(\frac{n-1}{2}\right) = \frac{7n-5}{2}$$

Case (ii) n is even:

Total number of operations

S= S_1 + ($S_2 + S_4 + ...S_n$)+($S_3 + S_5 + ...S_{n-1}$)

$$= 1+ 3\left(\frac{n}{2}\right) + 4\left(\frac{n-2}{2}\right) = \frac{7n-6}{2}$$

3.8.1. Comparing with no. of computations of LL^t decomposition of a symmetric, positive definite and tridiagonal matrix of order n.

S.No	Matrix		No.of arithmetical operations
1.	Symmetric, positive definite and tridiagonal matrix of order n		4n-3
2.	Symmetric, positive definite and tridiagonal matrix of order n in which (2k,2k-1) entries are zero.	n is even n is odd	(7n-6) / 2 (7n-5) / 2

CHAPTER – 4

Eigenvalues of Tridiagonal Matrices:

By definition a scalar $\lambda$is an eigenvalue of a square matrix A if there is a nonzero vector x such that $Ax=\lambda x$ and any such x is called eigenvector corresponding to the eigenvalue λ. Thus λ is an eigenvalue of A if the system of homogeneous equations represented by the matrix equation $(A-\lambda I)x = 0$. This system has a nontrivial solution if and only if the corresponding determinent $\det(A-\lambda I) = 0$.

If A has order n the above equation turns to be a polynomial equation of degree n and hence by the fundamental theorem of algebra has exactly n solutions. It is clear that A and its transpose A^t have the same eigenvalues and λ^{-1} is an eigervalue of A^{-1} whenever A is nonsingular. It is also true that if $A = S^{-1}BS$ then A and B have the same eigenvalues. That is, the eigenvalues remain invariant under a similarity transformation.

There are several methods for finding the eigenvalues of a matrix. When A is a real symmetric matrix Jacobi's method Given's method and Householder's method can be used to find the eigenvalues. A very effective method for finding the numerically largest eigenvalue and corresponding eigenvector is the power method.

When we have a problem involving an unsymmetrical (assymmetric) matrix we try to transform the unsymmetrical matrix into a symmetrical one having the same eigenvalues. For tridiagonal matrices there is a very handy and short way to make this transformation. Since a symmetric matrix has always real eigenvalues (proof supplied in [4.2]) this method can be useful to test if a general tridiagonal matrix has all real eigenvalues or not. The contents of this section are from [1].

4.1. Result [1]:

Given a unsymmetrical tridiagonal matrix A

$$A=\begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ c_2 & a_2 & b_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & c_3 & a_3 & b_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & c_4 & a_4 & b_4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & c_n & a_n \end{pmatrix}$$

Where c_k, b_{k-1} have same sign, assume that there exists a diagonal matrix D such that $S = D^{-1} AD$.

$S = D^{-1}AD \Rightarrow DS = AD$

Since A is tridiagonal S is tridiagonal

let $S=\begin{pmatrix} l_1 & s_1 & 0 & 0 & 0 & \dots & 0 \\ s_1 & l_2 & s_2 & 0 & 0 & \dots & 0 \\ 0 & s_2 & l_3 & s_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & s_{n-1} & l_n \end{pmatrix}$ and $D=\begin{pmatrix} d_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & d_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & d_n \end{pmatrix}$

DS = AD

$$\Rightarrow \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & d_n \end{pmatrix} \begin{pmatrix} l_1 & s_1 & 0 & 0 & \dots & 0 \\ s_1 & l_2 & s_2 & 0 & \dots & 0 \\ 0 & s_2 & l_3 & s_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & s_{n-1} & l_n \end{pmatrix}$$

$$
= \begin{pmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ c_2 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & c_3 & a_3 & b_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_n & a_n \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix} d_1 l_1 & d_1 s_1 & 0 & 0 & \dots & 0 \\ d_2 l_2 & d_2 l_2 & d_2 s_2 & 0 & \dots & 0 \\ 0 & d_3 s_2 & d_3 l_3 & d_3 s_s & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & d_n s_{n-1} & d_n l_n \end{pmatrix}
$$

$$
= \begin{pmatrix} a_1 d_1 & b_1 d_2 & 0 & 0 & \dots & 0 \\ c_2 d_1 & a_2 d_2 & b_2 d_3 & 0 & \dots & 0 \\ 0 & c_3 d_2 & a_3 d_3 & b_3 d_s & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & c_n d_{n-1} & a_n d_n \end{pmatrix}
$$

$$
\begin{array}{llll}
\Rightarrow \quad d_1 l_1 = a_1 d_1 & d_1 s_1 = b_1 d_2 & & \\
d_2 s_1 = c_2 d_1 & d_2 l_2 = a_2 d_2 & d_2 s_2 = b_2 d_3 & \\
0 & d_3 s_2 = c_3 d_2 & d_3 l_3 = a_3 d_3 & d_3 s_3 = b_3 d_4 \\
------------ & ------------- & ------------- & ------------- \\
------------ & ------------- & ------------- & ------------- \\
d_{n-1} s_{n-2} = c_{n-1} d_{n-2}, & d_{n-1} l_{n-1} = a_{n-1} d_{n-1}, & d_{n-1} s_{n-1} = b_{n-1} d_n & \\
d_n s_{n-1} = c_n d_{n-1} & d_n l_n = a_n l_n & &
\end{array}
$$

let $d_1 = 1 \Rightarrow l_1 = a_1, and \quad s_1 = b_1 d_2, \; d_2 s_1 = c_2$

$l_2 = a_2, \, l_3 = a_3, ..., l_n = a_n$

for i = 1(1)n-1

Case(i) $b_i = 0 \Rightarrow s_i = 0 \Rightarrow c_{i+1} = 0$

Case (ii) $b_i \neq 0 \Rightarrow c_{i+1} \neq 0$ because

$c_{i+1} = 0 \Rightarrow d_{i+1} s_i = 0 \Rightarrow s_i = 0 \Rightarrow b_i d_{i+1} = 0 \Rightarrow b_i = 0$ contradiction.

Let $b_i \neq 0, c_{i+1} \neq 0 \quad and \quad \frac{c_{i+1}}{b_i} > 0$

$d_i s_i = b_i d_{i+1}, d_{i+1} s_i = c_{i+d} d_i$

$\Rightarrow \frac{d_i}{d_{i+1}} = \frac{b_i d_{i+1}}{c_{i+1} d_i}$

$\Rightarrow d_{i+1}{}^2 = \frac{c_{i+1}}{b_i} d_i^2$

$\Rightarrow \frac{c_{i+1}}{b_i} > 0$ and $d_{i+1} = \sqrt{\frac{c_{i+1}}{b_i}}..d_i$

let $b_i \neq 0, c_{i+1} \neq 0 \; and \; \frac{c_{i+1}}{b_i} < 0$

$d_i s_i = b_i d_{i+1}, d_{i+1} s_i = c_{i+1} d_i$

$d_{i+1}{}^2 = \frac{c_{i+1}}{b_i} d_i^2 \Rightarrow \frac{c_{i+1}}{b_i} > 0$ contradiction.

Therefore the diagonal [d_1,d_2,d_3,……….d_n] can be obtained by the following iterative “square root” formula.

$d_1 = 1 \Rightarrow d_2 = \sqrt{\frac{c_2}{b_1}}.d_1 \Rightarrow d_3 = \sqrt{\frac{c_3}{b_2}}.d_2 .... \Rightarrow d_n = \sqrt{\frac{c_n}{b_{n-1}}}.d_{n-1}$

Condition for symmetrical tridiagonal transformation to have real eigenvalues:

This process can be applied if the square root argument is positive.

$$\frac{c_k}{b_{k-1}} > 0, k = 2(1)n$$

The above conditions are sufficient to transform the unsymmetrical tridiagonal matrix A into a symmetrical tridiagonal matrix S. The similarity transform doesn't alter the eigenvalues. Thus the matrices S and A have the same eigenvalues. Further the eigenvalues of S are real in view of the following.

4.2. Theorem[2]: The eigenvalues of a real symmetric matrix are real.

Butterfly test for real eigenvalues of tridiagonal matrix [1]:
As is mentioned above If each element couple of a tridiagonal matrix C_k, b_{k-1} (this couple, like a butterfly, is put in evidence in the figure) have the same sign, then the eigenvalues are real.

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_2 & a_2 & b_2 & 0 \\ 0 & c_3 & a_3 & b_3 \\ 0 & 0 & c_4 & a_4 \end{pmatrix}$$

We can say as well that the matrix can be converted in symmetrical form by a similarity transform, ie $S=D^{-1}AD$, where S is a symmetrical tridiagonal matrix.

Illustration: Consider the following 8 x 8 tridiagonal matrix,

$$A = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -7 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -16 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 8 & 18 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 27 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 \end{pmatrix}$$

Test : 4 x 1 > 0

1 x 4 > 0

2 x 8 > 0

(-16) x (-4) > 0

5 x 5 > 0

2 x 18 > 0

3 x 27 > 0

The butterfly test is positive. So the matrix can be converted into symmetric one by similar transform. With the iterative formula, we find the elements of diagonal matrix.

$$d_1 = 1 \Rightarrow d_2 = \sqrt{\frac{4}{1}}.1 = 2 \Rightarrow d_3 = \sqrt{\frac{1}{4}}.2 = 1..............$$

D = diag (1, 2, 1, 1/2 , 1, 1, 1/3 , 1/9)

D^{-1}= diag (1, ½, 1, 2, 1, 1, 3, 9)

Performing the similarity transform, $S = D^{-1}AD$

We get the symmetrical tridiagonal matrix

$$S = \begin{pmatrix} 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 6 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -7 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 8 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & -1 \end{pmatrix}$$

Breaking down method [1]:

When the tridiagonal matrix has one or more zero elements in the subdiagonal the above process cannot be applied. In this case the problem can be simplified breaking the given matrix into sub-matrices by partitioning.

Illustration:

$$\begin{pmatrix} 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 6 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -7 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & -1 \end{pmatrix}$$

$$\begin{array}{cc} \begin{pmatrix} 5 & 2 & 0 & 0 \\ 2 & 6 & 2 & 0 \\ 0 & 4 & -7 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} & 0 \\ 3 & \begin{pmatrix} -5 & 5 & 0 & 0 \\ 1 & 8 & 3 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 9 & -1 \end{pmatrix} \end{array}$$

$$B = \begin{pmatrix} 5 & 2 & 0 & 0 \\ 2 & 6 & 2 & 0 \\ 0 & 4 & -7 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} -5 & 5 & 0 & 0 \\ 1 & 8 & 3 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 9 & -1 \end{pmatrix}$$

For each of B,C we can now apply the process of similarity transform for finding the symmetrical form. Then, we find the eigenvalues of B,C separately. The union of these two sets of eigenvalues gives all the eigenvalues of the original matrix.

We consider symmetric, positive definite and tridiagonal matrix $A=(a_{ij})$ in which $a_{2k,2k-1}=0$.

Since matrix A is symmetric tridiagonal, breaking down method can be applied to find eigenvalues.

Illustration:

$$A = \begin{pmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & 7 & 4 & 0 & 0 \\ 0 & 4 & 9 & 0 & 0 \\ 0 & 0 & 0 & 10 & 3 \\ 0 & 0 & 0 & 3 & 4 \end{pmatrix}$$

Submatrices are

$S_1 = (15)$, $S_2 = \begin{pmatrix} 7 & 4 \\ 4 & 9 \end{pmatrix}$, $S_3 = \begin{pmatrix} 10 & 3 \\ 3 & 4 \end{pmatrix}$

Eigervalue of S_1 is $\lambda_1 = 15$

Eigenvalues of S_2 are $\lambda_2 = 12.12310563$ $\lambda_3 = 3.876894374$

Eigenvalues of S_3 are $\lambda_4 = 11.24264069$ $\lambda_5 = 2.757359313$

Therefore eigenvalues of A are

λ_1	15
λ_2	12.12310563
λ_3	11.24264069
λ_4	3.87689374
λ_5	2.757359313

SECTION – 5

QR Algorithm for Tridiagonal Matrices:

Definition: The matrix norm, ||A|| where A is a matrix of order n, is a non-negative number which satisfies the properties:

(i) ||A||>0 if A≠0 and ||0|| = 0

(ii) ||cA||= |c| ||A|| for an arbitrary complex number c

(iii) ||A+B|| ≤ ||A|| + ||B||

(iv) ||AB|| ≤ ||A|| ||B||

Definition: (Frobeneous) Euclidean norm: ||A|| = $(\sum_{i,j=1}^{n} | a_{ij} |^2)^{1/2}$

QR Decomposition :

5.1 Theorem [2] : If A is a matrix of order n there exits an orthogonal matrix Q and an upper triangular matrix R such that A = QR

Proof : step 1 :

For a matrix A = $(a_{ij}), let\ \ x_1 = (a_{11}, a_{21,} ......a_{n1})^t$

and y_1=(|| x_1 ||$_2$ 0,0,……0)^t, then there is a Householder matrix P_1 such that $P_1x_1=y_1$ and

$$P_1A = \begin{pmatrix} \| x_1 \|_2 & a_{12}^{(1)} & \ldots & a_{1n}^{(1)} \\ 0 & a_{22}^{(1)} & \ldots & a_{2n}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & a_{n2}^{(1)} & \ldots & a_{nn}^{(1)} \end{pmatrix}$$

Step 2: Let $x_2 = (0, a_{22}^{(1)}, ...a_{n2}^{(1)})^t\ and\ \ y_2 = (0, \| x_2 \|_2, 0, ...0)^t$ then as above there is a householder matrix P_2 such that $P_2 x_2 = y_2$ and

$$P_2P_1A = \begin{pmatrix} \|x_1\|_2 & a_{12}^{(1)} & a_{13}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & \|x_2\|_2 & a_{23}^{(2)} & \dots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3n}^{(2)} \\ \dots & \dots. & \dots. & \dots & \dots. \\ \dots & \dots. & \dots. & \dots & \dots. \\ 0 & 0 & a_{n3}^{(2)} & \dots & a_{nn}^{(2)} \end{pmatrix}$$

If we continue this procedure, we will have Householder matrices, P_3,……….., P_{n-1} such that

$$P_{n-1}P_{n-2}\dots\dots.P_2P_1A = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \dots & r_{1n} \\ 0 & r_{22} & r_{23} & \dots & r_{2n} \\ 0 & 0 & r_{33} & \dots & r_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & r_{nn} \end{pmatrix} = R$$ is an upper triangular matrix.

Since all P_I's are symmetric and orthogonal,

$P_I^{-1}=P_I^t=P_i$

Let Q = $P_{n-1}P_{n-2}$……………….P_2P_1 Then QA = R Therefore A = QR

Note: Q is orthogonal but not symmetric.

5.1. QR Algorithm[5]: Method of finding unitary matrix Q of order n and upper traingular matrix of order n for a matrix A = (a_{ij}) of order n.

Let a_k,q_k, and r_k be the kth column of A,Q,R respectively.

Step 1:

For k = 1 choose $r_{11} = \|a_1\|_2 = (a_{11}^2 + a_{21}^2 + \dots\dots\dots\dots.. + a_{n1}^2)^{1/2}$

$$q_1 = r_{11}^{-1}a_1$$

Step 2:

for k = 2(1)n

compute $r_{ik} = q_i^t a_k ,\ for\ \ i = 1(1)k-1$

$$r_{kk} = \| a_k - \sum_{i=1}^{k-1} r_{ik} q_i \|$$

$$q_k = r_{kk}^{-1}(a_k - \sum_{i=1}^{k-1} r_{ik} q_i)$$

Illustration:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

Step 1:

K=1, $r_{11} = (a_{11}^2 + a_{21}^2)^{1/2} = 2.236067977$

$$q_1 = r_{11}^{-1} a_1 = (0.447213595, 0.894427191, 0)^t$$

Step 2: K=2,

$$r_{12} = q_i^t a_2 = 3.577708764$$

$$r_{22} = \| a_2 - r_{12} q_1 \| = 4.024922359$$

$$q_2 = r_{22}^{-1}(a_2 - r_{12} q_1) = (0.099380799, -0.049690399, 0.99380799)^t$$

Step 3:K=3,

$$r_{13} = q_1^t a_3 = 3.577708764$$

$$r_{23} = q_2^t a_3 = 4.77062399$$

$$r_{33} = \| a_3 - r_{13} q_1 - r_{23} q_2 \| = 2.333333355$$

$$q_3 = r_{33}^{-1}(a_3 - r_{13} q_1 - r_{23} q_2)$$

= $(-0.8889036, 0.444451799, 0.110963896)^t$

$$Q = \begin{pmatrix} 0.447213595 & 0.099380799 & -0.8889036 \\ 0.894427191 & -0.049690399 & 0.444451799 \\ 0 & 0.99380799 & 0.110963896 \end{pmatrix}$$

$$R = \begin{pmatrix} 2.236067977 & 3.577708763 & 3.577708764 \\ 0 & 4.024922359 & 4.77062399 \\ 0 & 0 & 2.333333355 \end{pmatrix}$$

$$QR = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \\ 0 & 4 & 5 \end{pmatrix} = A$$

We examine how the QR algorithm gets simplified for TD matrices.

5.2. QR factorization for tridiagonal matrices:

Let A=(a_{ij}) be tridiagonal matrix of order n.

So, a_{ij} = 0 for |i-j| >1

and A = QR where Q is unitary and R is upper traingular matrix.

let Q = (q_{ij}) and R =(r_{ij}) where $r_{ij} = 0$ for i-j ≥1

from this of section 5

$$q_{ij} = 0 \quad for \; i - j \geq 2$$

We have $a_{ij} = 0$ for i - j ≥ 2

So

$$q_j = r_{jj}^{-1}(a_j - \sum_{i=1}^{j-1} r_{ij} q_i)$$

$$= r_{jj}^{-1}\left[\begin{pmatrix}0\\0\\ \\ a_{j-1,j}\\ a_{jj}\\ a_{j+1,j}\\ 0\\ 0\end{pmatrix} - r_{1j}\begin{pmatrix}p_{11}\\ p_{21}\\ 0\\ \\ \\ \\ \\ 0\end{pmatrix} - r_{2j}\begin{pmatrix}p_{12}\\ p_{22}\\ p_{32}\\ 0\\ \\ \\ \\ 0\end{pmatrix} \;.....\; r_{j-1,j}\begin{pmatrix}p_{1,j-1}\\ p_{2,j-1}\\ .\\ \\ p_{j,j-1}\\ 0\\ \\ 0\end{pmatrix}\right]$$

where $p_{i1} = r_{11}^{-1} a_{i1},\ for\ \ i = 1(1)n$

$p_{ij} = r_{jj}^{-1}(a_{ij} - r_{ij}\, p_{i,j-1}),\ for\ \ j = 2(1)n$

for k ≥ j+2, the k^{th} term in each of the column is zero.

Hence the k^{th} term of q_j, is zero.

This is true ∀ k ≥ j+2 and ∀ j ≥ 1

∴ Q is upper Hessenberg.

We now present our results on QR factorization of a matrix in SLTD.

5.3. Result:

If A is symmetric, positive definite and tridiagonal matrix of order n in which $a_{2k,2k-1}$=0 ∀ k then the matrix Q in QR factorization is also of the same type as A and in the upper triangular matrix R, $r_{2i-1,2i}$ =0 and r_{ij}=0 for i-j ≥ 1 and j –i ≥2.

Proof: k=1

$r_{11} = \| a_1 \| = a_{11}$

$q_1 = r_{11}^{-1} a_1 = (p_{11} 0\ \ 0.....0)^t$ where

$p_{i1} = r_{11}^{-1} a_{i1}\ \ for\ \ i = 1(1)n$

$p_{ij} = r_{jj}(a_{ij} - r_{ij}\, p_{i,j-1})\ for\ \ j = 2(1)n$

k = 2

$r_{12} = q_1^t a_2 = (p_{11} 0\ \ 0........0)(0\ \ \ a_{22}\ \ a_{32}\ \ \ 0..........0)^t$ =0

$r_{22} = \| a_2 - r_{12} q_1 \| = \| a_2 \|$

$= (a_{22}^2 + a_{32}^2)^{1/2}$

$q_2 = r_{22}^{-1}(a_2 - r_{12}q_1) = r_{22}^{-1}a_2$

$=(0 \ \ p_{22}\, p_{32} \ \ 0 \quad 0)^t$

for i =1(1)n/2, when n is even

= 1(1) (n-1/2), when n is odd

k = 2i

$r_{1,2i} = q_1^t a_{2i} = (p_{11} 0 \ \ 0.....0)(0 \ \ 0......a_{2i,2i}, a_{2i+1,2i}, 0 \ \ 0,.....0)^t = 0$

for j = 2(1) 2i-1

Assume $q_j = (00.........p_{jj}\, p_{j+1,j}.....0)^t, if \ \ j \ \ is \ even$

$= (00.........p_{j-1j}\, p_{jj}.....0)^t, if \ \ j \ \ is \ odd$

(i) j is even

$r_{j,2i} = q_j^t a_{2i}$

$= (00........op_{jj}\, p_{j+1,j} 0....0)(0 \ 0.....a_{2i,2i}, a_{2i+1,2i}....0)^t$

$= p_{jj}(0) + p_{j+1,j}(0) + (0)a_{2i,2i} + (0)a_{2i+1,2i} = 0$

(ii) j is odd

$r_{j,2i} = q_j^t a_{2i}$

$= (00........p_{j-1,j}\, p_{jj} 0....0)(0 \ 0.....a_{2i,2i}, a_{2i+1,2i}....0)^t$

$= p_{j-1,j}(0) + p_{jj}(0) + (0)a_{2i,2i} + (0)a_{2i+1,2i}$

= 0

$\therefore \ r_{j,2i} = 0$ for j=1(1)2i-1

$$r_{2i,2i} = \| a_{2i} - \sum_{j=1}^{2i-1} r_{j,2i} q_j \|$$

$= \| a_{2i} \|$

$q_{2i} = r_{2i,2i}^{-1} a_{2i}$

$= (0\ 0......p_{2i,2i}, p_{2i+1,2i} \quad 0......0)^t$

$K = 2i+1,$

$r_{1,2i+1} = q_1^t a_{2i+1}$

$= (p_{11}\ 0 \quad 0.....0)\ (0 \quad 0....a_{2i,2i+1}, a_{2i+1,2i+1}.....0)^t = 0$

for j = 2(1) 2i-1

j is even,

$r_{j,2i+1} = q_j^t a_{2i+1}$

$= (0\ \ 0....p_{jj}, p_{j+1,j} \ \ 0....0)(00...a_{2i,2i+1}, a_{2i+1,2i+1},......0)^t$

$= p_{jj}(0) + p_{j+1,j}(0) + (0)a_{2i,2i+1} + (0)a_{2i+1,2i+1} = 0$

j is odd

$r_{j,2i+1} = q_j^t a_{2i+1}$

$= (0\ \ 0....p_{j-1,j}, p_{jj,} \ \ 0....0)(00...a_{2i,2i+1}, a_{2i+1,2i+1},......0)^t$

$= p_{j-1,j}(0) + p_{jj}(0) + (0)a_{2i,2i+1} + (0)a_{2i+1,2i+1} = 0$

$\therefore r_{j,2i+1} = 0 \; for \quad j = 2(1)2i-1$

$r_{2i,2i+1} = q^t{}_{2i} \quad a_{2i+1}$

$= (0 \quad 0.....p_{2i,2i}\, p_{2i+1,2i}....0)(0...a_{2i.2i+1}, a_{2i+1,2i+1}......0)^t$

$= p_{2i,2i} a_{2i.2i+1} + p_{2i+1,2i} \quad a_{2i+1,2i+1} \neq 0$

$$r_{2i+1,2i+1} = \| a_{2i+1} - \sum_{j=1}^{2i} r_{j,2i+1} \quad q_j \|$$

$= \| a_{2i+1} - r_{2i,2i+1} \quad q_{2i} \| \neq \; 0$

$q_{2i+1} = r^{-1}_{2i+1,2i+1}(a_{2i+1} - r_{2i,2i+1} \quad q_{2i})$

$= (0 \;\; 0......p_{2i,2i+1}, p_{2i+1,2i+1},......0)^t$

QR = A

$$\text{where } A = \begin{pmatrix} p_{11} & 0 & 0 & ... & 0 \\ 0 & p_{22} & p_{23} & ... & 0 \\ 0 & p_{32} & p_{33} & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & p_{n,n-1} & p_{nn} \end{pmatrix}$$

$$\text{and} \quad R = \begin{pmatrix} r_{11} & 0 & 0 & 0 & 0 & ... & 0 \\ 0 & r_{22} & r_{23} & 0 & 0 & ... & 0 \\ 0 & 0 & r_{33} & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & r_{44} & r_{45} & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & ... & r_{nn} \end{pmatrix}$$

REFERENCES

1. **Foxes Team** : Numeric Calculus, free document from Internet.

2. **Golub H. Gene, Charles F. Van Loan** : Matrix Computations, North Oxford Academic, Oxford

3. **Jain, M.K. and Chawla, M.M**. : Numerical Analysis For Scientists and Engineers, SBW publishers, (Delhi) 1971

4. **Jain,M.K., Iyengar, S.R.K.; Jain, R.K**. : Numerical Methods for Scientific and Engineering Computations, Wiley Eastern Limited, New Delhi.

5. **Krishna Murthy, E.V. ; Sen, S.K.** ; Numerical Algorithms,
Computations in Science and Engineering, Affiliated East- West Press Pvt Ltd. , New Delhi.

6. **Rami Reddy,B.** : On The Inverse of a Nonsingular Matrix And The Eigenvalues of a Matrix.

7. **Rami Reddy,B**. : Matrix Computations and Applications to Operator Theory.